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Higher relative index theorems for foliations

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ABSTRACT

In this paper we solve the general case of the cohomological relative index problem for foliations of non-compact manifolds. In particular, we significantly generalize the groundbreaking results of Gromov and Lawson, [20], to Dirac operators defined along the leaves of foliations of non-compact complete Riemannian manifolds, by involving all the terms of the Connes-Chern character, especially the higher order terms in Haefliger cohomology. The zero-th order term corresponding to holonomy invariant measures was carried out in [8] and becomes a special case of our main results here. In particular, for two leafwise Dirac operators on two foliated manifolds which agree near infinity, we define a relative topological index and the Connes-Chern character of a relative analytic index, both being in relative Haefliger cohomology. We show that these are equal. This invariant can be paired with closed holonomy invariant currents (which agree near infinity) to produce higher relative scalar invariants. When we relate these invariants to the leafwise index bundles, we restrict to Riemannian foliations on manifolds of sub-exponential growth. This allows us to prove a higher relative index bundle theorem, extending the classical index bundle theorem of [5]. Finally, we construct examples of foliations and use these invariants to prove that their spaces of leafwise positive scalar curvature metrics have infinitely many path-connected components, completely new results which are not available

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from [8]. In particular, these results confirm the well-known idea that important geometric information of foliations is embodied in the higher terms of the $\widehat{A}$ genus.

Contents

1. Introduction

In this paper we continue our program of extending the groundbreaking relative index theorems of Gromov-Lawson, especially Theorem 4.18, [20], to Dirac operators defined along the leaves of foliations of non-compact complete Riemannian manifolds. Their results have played a fundamental role in the development and understanding of the existence and non-existence of metrics with positive scalar curvature (PSC), as well as the structure of spaces of such metrics. It is an essential tool for the extension of results for compact manifolds to non-compact manifolds.

In [8], we extended the Gromov-Lawson theorem to foliations admitting invariant transverse measures, and crucial requirements for the applications were that the foliation admits a holonomy invariant measure, and that the measured $\widehat{A}$ genus of the foliation be non-zero. In this paper, we dispense with both these requirements and completely solve the general case. We obtain results for all the terms of the Atiyah-Singer characteristic forms associated with the Dirac operators, especially the higher order terms of the Connes-Chern character of the relative analytic index, as well as the higher order terms of the Connes-Chern characters of their "index bundles". We also construct a large collection of spin foliations, with trivial zero-th order Haefliger $\widehat{A}$ genus, whose spaces of leafwise PSC metrics have infinitely many path connected components. In particular, these results confirm the idea that the higher order terms of the $\widehat{A}$ genus carry important geometric information.

As in [8], our work is in the spirit of the transition from the Atiyah-Singer index theorem, [2], to Connes' index theorem for foliations, [13–15]. In order to overcome the problems of dealing with non-compact manifolds, we assume that our objects have bounded geometry. Our higher relative index theorem then provides the expected formula in an appropriate relative Haefliger cohomology for pairs of foliations which are isomorphic near infinity, equating the higher relative analytical index constructed out of parametrics with the higher relative A-hat forms. When the foliations are top dimensional, we recover the Gromov-Lawson theory [20,27]. When the foliated manifolds are compact (without boundary), we recover the cohomological version of the Connes-Skandalis index theorem [15], as developed in [4] using Haefliger cohomology. When the foliations are not top-dimensional, any pair of Haefliger transverse currents which are compatible near infinity lead to scalar higher relative index formulas. We thus recover the results of our previous paper [8] by pairing our higher relative index formula with a compatible pair of holonomy invariant transverse measures.

As is well known and already observed for closed foliated manifolds, see for instance [5,6,25], despite the top-dimensional case, further conditions are required to relate the higher analytic index of leafwise Dirac operators to their spectral index, say the Connes-Chern characters of the leafwise projections to their kernels, the so-called index bundle. The examples in [6] show that such restrictions are necessary. Assuming, as in [5], that the spectral projections of the leafwise Dirac operators are sufficiently sparse near zero and that the foliations are Riemannian, we prove our next higher index theorem which now involves the relative spectral index. This theorem holds only in the absolute Haefliger cohomologies since the pair of index bundles is in general not compatible near infinity. This incompatibility can prevent the pairing of the index bundles with compatible near infinity Haefliger currents from being well defined. Finally, we show that when the ambient manifolds have sub-exponential growth, such pairings are miraculously well defined as soon as the Dirac operators are invertible near infinity, and they equal the pairing with the higher relative A-hat forms. The invertibility near infinity is the usual Gromov-Lawson condition involving the zero-th order term of the Bochner formula. It occurs for instance when the foliations are spin with leafwise PSC near infinity, compare with [20].

Notational details are given in the next section.

Denote by (M, F) a foliated manifold where M is a non-compact complete Riemannian manifold and F is an oriented foliation (with the induced metric) of M. We assume that both M and F are of bounded geometry and that the holonomy groupoid of F is Hausdorff. We will sometimes assume that F is Riemannian, and when we do, we will explicitly point it out in the text. The general case will be addressed in [9]. We assume that we have a Clifford bundle $E_M \to M$ over the Clifford algebra of the cotangent bundle to F, along with a Hermitian connection $\nabla^{F,E}$ compatible with Clifford multiplication. This determines a leafwise generalized Dirac operator, denoted D_F. We assume that we have a second foliated manifold (M', F') with the same structures. We further assume that there are compact subspaces $\mathcal{K}_M = M \setminus V_M$ and $\mathcal{K}'_{M'} = M' \setminus V'_{M'}$

so that the situations on V_M and $V'_{M'}$ are identical via a smooth isometry φ. These are the usual Gromov-Lawson relative data. Note that in our case, the "bad set" restricted to a leaf need *not* be compact as in the Gromov-Lawson case. Only the global aggregate of all such leafwise subsets needs to be compact as a subset of M.

In [8], we worked on the ambient manifolds M and M'. Here we work on their holonomy groupoids $\mathcal{G}$ and $\mathcal{G}'$, with their canonical foliations F_s and F'_s, as we did in [5]. We lift everything to $\mathcal{G}$ using the range map $r : \mathcal{G} \to M$, which is a covering map from the leaves of F_s to those of F, and similarly for M'. In particular, we have the $\mathcal{G}$ invariant leafwise Dirac operator D for the foliation F_s, and similarly D' for F'_s.

Recall that for a good cover $\mathcal{U} = \{(U_i, T_i)\}$ of M, [24], by foliation charts U_i with local complete transversals $T_i \subset U_i$, the Haefliger forms associated to F are the bounded smooth differential forms on $\amalg T_i$ which have compact support in each T_i, modulo forms minus their holonomy images. The (absolute) Haefliger cohomology of F, denoted $H^*_c(M/F)$, is then the associated de Rham cohomology, and is independent of the choice of good cover, [21]. Also recall that there is an *integration over the leaves* map from forms on M to Haefliger forms, denoted $\int_F$, which induces a map on cohomology. For the foliation given by the fibers of a bundle $M \to B$, the Haefliger cohomology reduces to the cohomology of the base and $\int_F$ is the classical integration over the fibers map. See again [21] for more details.

The receptacle for our relative index formulas will be a relative version of Haefliger cohomology that we denote by $H^*_c(M/F, M'/F'; \varphi)$. This is the cohomology of pairs of Haefliger forms which agree near infinity (that is, on T_i far enough away from $\mathcal{K}_M$ and similarly for the T'_i), again modulo pairs of forms minus their holonomy images which also agree near infinity.

Denote by $AS(D_F)$ the Atiyah-Singer characteristic differential form, associated with the above φ-compatible data, for D_F, and similarly for $D_{F'}$. These differential forms agree near infinity on M and M'. The relative A-hat genus of the compatible pair (D, D'), alternatively called the relative topological index, is

$$\mathrm{Ind}_t(D, D') \;=\; \left[\int_F \mathrm{AS}(D_F), \int_{F'} AS(D'_{F'})) \right] \;\in\; H^*_c(M/F, M'/F'; \varphi).$$

Using parametrics, we define a relative analytical index class $\mathrm{Ind}_a(D, D')$ in the appropriate K-theory group, and its Connes-Chern character,

$$\mathrm{ch}(\mathrm{Ind}_a(D, D')) \;\in\; H^*_c(M/F, M'/F'; \varphi).$$

Our first result is

Theorem 4.2. *For the (M, F), (M', F'), D and D' as above,*

$$\mathrm{ch}(\mathrm{Ind}_a(D, D')) = \mathrm{Ind}_t(D, D') \ \text{ in } H_c^*(M/F, M'/F'; \varphi).$$

So, pairing with any compatible near infinity pair (C, C') of closed Haefliger currents yields a (higher) scalar relative index formula. Such pairings will be denoted $\langle \cdot, \cdot \rangle$, e.g. $\langle \mathrm{ch}(\mathrm{Ind}_a(D, D')), (C, C') \rangle$.

An important application of this theorem is to pairs of "reflective" foliations, which we consider in Section 6. They can be "cut and pasted" to get a compact foliated manifold $\widehat{M}$, with the foliation $\widehat{F}$ and operator $\widehat{D}_{\widehat{F}}$. Given C and C' as above, denote by $\widehat{C}$ the current they determine on $\widehat{M}$. Then we have the following extension of the Gromov-Lawson Relative Index Theorem, see [20], which is most useful in Section 7, where we construct our examples.

Theorem 6.7. *Suppose that F (and so also F') is reflective. Then*

$$\langle \mathrm{ch}(\mathrm{Ind}_a(D, D'), (C, C') \rangle \ = \ \langle \mathrm{ch}(\mathrm{Ind}_a(\widehat{D}_{\widehat{F}})), \widehat{C} \rangle.$$

The RHS of this index formula can then be computed using the classical higher cohomological index theorem for foliations of closed manifolds [17,4]. For top dimensional foliations, say when $TF = TM$ and $TF' = TM'$, the previous two theorems reduce to the classical Gromov-Lawson relative index theorems.

Despite the top dimensional case, it is well known that the higher index is not easily related with the so-called index bundle, i.e. the Chern character of the "kernel minus cokernel superbundle". Constraints on the spectral distributions, as well as on the geometry near infinity are necessary, see for instance [6]. Denote by P_0 the leafwise spectral projection to the kernel of D^2. In general P_0 is not transversely smooth (although it is always leafwise smooth), and if not, we cannot even define its Connes-Chern character in our Haefliger cohomology without perturbing the operator. There are though interesting classes of foliations and leafwise Dirac-type operators whose kernel superbundle P_0 is transversely smooth, and in this case, we get a well defined spectral index class

$$\mathrm{ch}(P_0) \ \in \ H_c^*(M/F),$$

and similarly for P_0', see [5].

Denote by $P_{(0,\epsilon)}$ the leafwise spectral projection for D^2 for the interval $(0, \epsilon)$. The Novikov-Shubin invariants $NS(D)$ of D are a measure of the density of the image of $P_{(0,\epsilon)}$. The larger $NS(D)$ is, the sparser the image of $P_{(0,\epsilon)}$ is as $\epsilon \to 0$.

We also have the natural map $(\pi \times \pi') : H_c^*(M/F, M'/F'; \varphi) \to H_c^*(M/F) \times H_c^*(M'/F')$, and with it the Riemannian Foliation Relative Index Bundle Theorem.

Theorem 4.3. *Fix $0 \leq \ell \leq q/2$, where q is the codimension of F and F'. Assume that:*

- *the foliations F and F' are Riemannian;*
- *the leafwise operators P_0, P'_0, $P_{(0,\epsilon)}$ and $P'_{(0,\epsilon)}$ (for ϵ sufficiently small) are transversely smooth;*
- *$NS(D)$ and $NS(D')$ are greater than ℓ.*

Then, for $0 \leq k \leq \ell$, we have in $H_c^{2k}(M/F) \times H_c^{2k}(M'/F')$

$$(\pi \times \pi') \operatorname{ch}^k(\operatorname{Ind}_a(D, D')) \;=\; (\operatorname{ch}^k(\operatorname{Ind}_a(D)), \operatorname{ch}^k(\operatorname{Ind}_a(D'))) \;=\; (\operatorname{ch}^k(P_0), \operatorname{ch}^k(P'_0)).$$

For Riemannian foliations, important examples of compatible near infinity pairs of closed Haefliger currents are given by closed bounded holonomy invariant transverse differential forms ω on M and ω' on M' which agree near infinity. These determine closed bounded Haefliger forms on T, denoted ω_T and $\omega'_{T'}$ which agree near infinity. Denote by dx the global volume form on M.

We then have the Higher Relative Index Pairing Theorem.

Theorem 4.6. *In addition to the assumptions in Theorem 4.3, assume that for ϵ sufficiently small, $\int_M \operatorname{tr}(P_{[0,\epsilon)})dx < \infty$ and $\int_{M'} \operatorname{tr}(P'_{[0,\epsilon)})dx < \infty$, and that M, and so also M', has sub-exponential growth. Then, for any $\omega \in C^\infty(\wedge^{q-2k}\nu^*)$ and $\omega' \in C^\infty(\wedge^{q-2k}\nu'^*)$ $(0 \leq k \leq \ell)$ as above,*

$$\int_T \operatorname{ch}(P_0) \wedge \omega_T \text{ and } \int_{T'} \operatorname{ch}(P'_0) \wedge \omega'_{T'} \text{ are well defined complex numbers,}$$

and

$$\int_T \operatorname{ch}(P_0) \wedge \omega_T \;-\; \int_{T'} \operatorname{ch}(P'_0) \wedge \omega'_{T'} \;=\; \langle \left[\int_F \operatorname{AS}(D_F), \int_{F'} AS(D'_{F'}) \right], [\omega_T, \omega'_{T'}] \rangle.$$

In Section 6, we show that the finite integral assumptions in Theorem 4.6 are satisfied when D_F (and hence also $D_{F'}$) is invertible near infinity, i.e. when the zeroth order differential operator $\mathcal{R}_F^E$ in the associated Bochner Identity

$$D_F^2 = \nabla^*\nabla + \mathcal{R}_F^E,$$

is uniformly positive near infinity on M. The sub-exponential growth condition can be extended to exponential growth provided it is not too robust. See Remark 5.7.

For a single foliated manifold with a pair of compatible near infinity leafwise Dirac operators, we have the following generalization of a classical result of Gromov-Lawson [20], compare with Theorem 6.5 in [27].

Theorem 6.5. *Suppose that E and E' are two Clifford bundles over the foliated manifold (M, F), which are isomorphic off the compact subset $\mathcal{K}_M$, with associated twisted Dirac operators D and D'. Let ω be a bounded closed holonomy invariant transverse form (or Haefliger current) of degree $\ell \leq q$. Suppose that*

- *M has sub-exponential growth, and F is Riemannian;*
- *the leafwise operators P_0, P'_0, $P_{(0,\epsilon)}$ and $P'_{(0,\epsilon)}$ (for ϵ sufficiently small) are transversely smooth;*
- *$\min(NS(D), NS(D'))$ is greater than ℓ;*
- *$\mathcal{R}^E_F$, and hence also $\mathcal{R}^{E'}_F$, is uniformly positive near infinity.*

Then, since $\mathrm{ch}(E) = \mathrm{ch}(E')$ off $\mathcal{K}_M$,

$$\int_{\mathcal{K}_M} (\mathrm{AS}(D_F)(\mathrm{ch}(E) - \mathrm{ch}(E')) \wedge \omega \;=\; \int_T (\mathrm{ch}(P_0) \;-\; \mathrm{ch}(P'_0)) \wedge \omega_T.$$

In the reflective case, again more constraints are necessary to obtain the link with the index bundle, and we have the following.

Theorem 6.8. *Suppose that F (so also F') is reflective. Suppose furthermore that $\widehat{F}$ is Riemannian and that $\widehat{P}_0$ and $\widehat{P}_{(0,\epsilon)}$ are transversely smooth and the Novikov-Shubin invariants of $\widehat{D}_{\widehat{F}}$ are greater than ℓ, for some $0 \leq \ell \leq q/2$. Then for any 2ℓ homogeneous φ-compatible (ω, ω') as above,*

$$\langle \mathrm{ch}(\mathrm{Ind}_a(D, D')), [\omega_T, \omega'_{T'}] \rangle = \langle \mathrm{ch}(\widehat{P}_0), \widehat{\omega}_{\widehat{T}} \rangle.$$

Moreover, if we impose the assumptions of Theorem 4.6, then

$$\langle (\mathrm{ch}(P_0), \mathrm{ch}(P'_0)), (\omega_T, \omega'_{T'}) \rangle \;=\; \langle \mathrm{ch}(\widehat{P}_0), \widehat{\omega}_{\widehat{T}} \rangle.$$

In Section 7, we consider foliations which admit positive scalar curvature (PSC) leafwise metrics. Given such a foliation, we associate to any pair (g_0, g_1) of such metrics, an invariant living in Haefliger cohomology, which provides an obstruction for the leafwise path connected equivalence of g_0 and g_1. This precisely generalizes the classical Gromov-Lawson invariant. Finally, we construct a large collection of spin foliations whose space of leafwise PSC metrics has infinitely many path connected components.

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2. The setup

In order that this paper be self contained, we begin this section recalling some material from [8].

Denote by M a smooth non-compact complete Riemannian manifold of dimension n, and by F an oriented foliation (with the induced metric) of M of dimension p, (until further notice, we assume that p is even), and codimension $q = n - p$. The tangent and cotangent bundles of M and F are denoted TM, T^*M, TF and T^*F. The normal and dual normal bundles of F are denoted ν and ν^*. A leaf of F is denoted by L. At times, we will assume that F is Riemannian, that is the metric on M, when restricted to ν is bundle like, so the holonomy maps of ν and ν^* are isometries. We will consider the general case in [9].

We assume that both M and F are of bounded geometry, that is, the injectivity radius on M and on all the leaves of F is bounded below, and the curvatures and all of their covariant derivatives on M and on all the leaves of F are bounded (the bound may depend on the order of the derivative).

Let $\mathcal{U}$ be a good cover of M by foliation charts as defined in [24]. In particular, denote by $\mathbb{D}^p(r) = \{x \in \mathbb{R}^p, ||x|| < r\}$, and similarly for $\mathbb{D}^q(r)$. An open uniformly locally finite cover $\{(U_i, \psi_i)\}$ of M by foliation coordinate charts $\psi_i : U_i \to \mathbb{D}^p(1) \times \mathbb{D}^q(1) \subset \mathbb{R}^n$ is a good cover for F provided that

1. For each $y \in \mathbb{D}^q(1), P_y = \psi_i^{-1}(\mathbb{D}^p(1) \times \{y\})$ is contained in a leaf of F. P_y is called a plaque of F.
2. If $U_i \cap U_j \neq \emptyset$, then it is contractible, and if $\overline{U}_i \cap \overline{U}_j \neq \emptyset$, then $U_i \cap U_j \neq \emptyset$.
3. Each ψ_i extends to a diffeomorphism $\psi_i : V_i \to \mathbb{D}^p(2) \times \mathbb{D}^q(2)$, so that the cover $\{(V_i, \psi_i)\}$ satisfies (1) and (2), with $\mathbb{D}^p(1)$ and $\mathbb{D}^q(1)$ replaced by $\mathbb{D}^p(2)$ and $\mathbb{D}^q(2)$.
4. Each plaque of V_i intersects at most one plaque of V_j and a plaque of U_i intersects a plaque of U_j if and only if the corresponding plaques of V_i and V_j intersect.
5. There are global positive upper and lower bounds on the norms of each of the derivatives of the ψ_i.

Bounded geometry foliated manifolds always admit good covers. In particular, we may assume that the V_i have a uniform bound on their diameters significantly less than the bound on the injectivity radii.

For each $U_i \in \mathcal{U}$, let $T_i \subset U_i$ be a local complete transversal (e.g. $T_i = \psi_i^{-1}(\{0\} \times \mathbb{D}^q(1))$) and set $T = \bigcup T_i$. We may assume that the closures of the T_i are disjoint. Given (U_i, T_i) and (U_j, T_j), suppose that $\gamma_{ij\ell} : [0, 1] \to M$ is a path whose image is contained in a leaf with $\gamma_{ij\ell}(0) \in T_i$ and $\gamma_{ij\ell}(1) \in T_j$. Then $\gamma_{ij\ell}$ induces a local diffeomorphism $h_{\gamma_{ij\ell}} : T_i \to T_j$, with domain $\mathrm{Dom}_{\gamma_{ij\ell}}$ and range $\mathrm{Ran}_{\gamma_{ij\ell}}$, which are assumed to be maximal. Note that the domains of the $h_{ij\ell}$ cover T, as do the ranges. The space $\mathcal{A}_c^k(T)$ consists of all smooth k-forms on T which are C^∞ bounded and have compact support in each T_i. The Haefliger k-forms for F, denoted $\mathcal{A}_c^k(M/F)$, consists of elements in the

quotient of $\mathcal{A}_c^k(T)$ by the closure of the vector subspace $\mathcal{W}$ generated by the elements of the form $\alpha_{ij\ell} - h^*_{\gamma_{ij\ell}}\alpha_{ij\ell}$ where $\alpha_{ij\ell} \in \mathcal{A}_c^k(T)$ has support contained in $\mathrm{Ran}_{\gamma_{ij\ell}}$. We need to take care as to what this means. Members of $\mathcal{W}$ consist of possibly infinite sums of elements of the form $\alpha_{ij\ell} - h^*_{\gamma_{ij\ell}}\alpha_{ij\ell}$, with the restriction that each i and each j appears at most a finite number of times. The projection map is denoted

$$[\cdot] : \mathcal{A}_c^*(T) \ \to \ \mathcal{A}_c^*(M/F).$$

Denote the exterior derivative by $d_T : \mathcal{A}_c^k(T) \to \mathcal{A}_c^{k+1}(T)$, which induces $d_H : \mathcal{A}_c^k(M/F) \to \mathcal{A}_c^{k+1}(M/F)$. Note that $\mathcal{A}_c^k(M/F)$ and d_H are independent of the choice of cover $\mathcal{U}$. The cohomology $H_c^*(M/F)$ of the complex $\{\mathcal{A}_c^*(M/F), d_H\}$ is the Haefliger cohomology of F.

Denote by $\mathcal{A}_u^*(M)$ the space of differential forms on M which are smooth and C^∞ bounded, and denote its exterior derivative by d_M and its cohomology by $H_u^*(M;\mathbb{R})$. As the bundle TF is oriented, there is a continuous open surjective linear map, called integration over F,

$$\int_F : \mathcal{A}_u^{p+k}(M) \to \mathcal{A}_c^k(T),$$

which commutes with the exterior derivatives. This map is given by choosing a partition of unity $\{\phi_i\}$ subordinate to the cover $\mathcal{U}$, and setting $\int_F \omega$ to be the class of $\sum_i \int_{U_i} \phi_i\omega$. It is a standard result, [21], that the image of this differential form $\left[\int_F \omega\right] \in \mathcal{A}_c^k(M/F)$ is independent of the partition of unity and of the cover $\mathcal{U}$. As $\int_F$ commutes with d_M and d_H, it induces the map $\int_F : H_u^{p+k}(M;\mathbb{R}) \to H_c^k(M/F)$.

Note that $\int_{U_i}$ is integration over the fibers of the projection $U_i \to T_i$, and that each integration $\omega \to \int_{U_i} \phi_i\omega$ is essentially integration over a compact fibration, so $\int_F$ satisfies the Dominated Convergence Theorem on each $U_i \in \mathcal{U}$.

The holonomy groupoid $\mathcal{G}$ of F consists of equivalence classes of paths $\gamma : [0,1] \to M$ such that the image of γ is contained in a leaf of F. Two such paths γ_1 and γ_2 are equivalent if $\gamma_1(0) = \gamma_2(0)$, $\gamma_1(1) = \gamma_2(1)$, and the holonomy germ along them is the same. Two classes may be composed if the first ends where the second begins, and the composition is just the juxtaposition of the two paths. This makes $\mathcal{G}$ a groupoid. The space $\mathcal{G}^{(0)}$ of units of $\mathcal{G}$ consists of the equivalence classes of the constant paths, and we identify $\mathcal{G}^{(0)}$ with M.

The basic open sets defining the (in general non-Hausdorff) $2p+q$ dimensional manifold structure of $\mathcal{G}$ are given as follows. Given $U_i, U_j \in \mathcal{U}$ and a leafwise path $\gamma_{ij\ell}$ starting in U_i and ending in U_j, define the graph chart $U_i \times_{\gamma_{ij\ell}} U_j$ to be the set of equivalence classes of leafwise paths starting in U_i and ending in U_j which are homotopic to $\gamma_{ij\ell}$ through a homotopy of leafwise paths whose end points remain in U_i and U_j respectively. It is easy to see, using the holonomy map $h_{\gamma_{ij\ell}} : T_i \to T_j$ that $U_i \times_{\gamma_{ij\ell}} U_j \simeq \mathbb{D}^p(1) \times \mathbb{D}^p(1) \times \mathbb{D}^q(1)$.

$\mathcal{G}$ has the natural maps $r, s : \mathcal{G} \to M$, with $s([\gamma]) = \gamma(0)$ and $r([\gamma]) = \gamma(1)$. It also has two natural foliations, F_s and F_r, whose leaves are the fibers of s and r. We will primarily use F_s, whose leaves are denoted $\widetilde{L}_x = s^{-1}(x)$, for $x \in M$. Note that $r : \widetilde{L}_x \to L$ is the holonomy covering map. We will assume that $\mathcal{G}$ is Hausdorff, which is automatic for Riemannian foliations.

The smooth sections of a bundle E are denoted by $C^\infty(E)$, and those with compact support by $C^\infty_c(E)$. We assume that any connection or any metric on E, and all their derivatives, are bounded. See [30] for material about bounded geometry bundles and their properties.

For a real or complex bundle $E_M \to M$, the external tensor product bundle $E_M \boxtimes E_M^* \to M \times M$ can be pulled back under (s, r) to a smooth bundle denoted $E \boxtimes E^*$ over $\mathcal{G}$. We denote the smooth, bounded sections $k(\gamma)$ with compact support of the restriction of this bundle to the subset $U_i \times_{\gamma_{ij\ell}} U_j \subset \mathcal{G}$ by $C^\infty_c(U_i \times_{\gamma_{ij\ell}} U_j, E \boxtimes E^*)$. We extend them to all of $\mathcal{G}$ by setting $k(\gamma) = 0$ if $\gamma \notin U_i \times_{\gamma_{ij\ell}} U_j$.

Definition 2.1. [7] The algebra $C^\infty_u(E \boxtimes E^*)$ consists of smooth sections k of $E \boxtimes E^*$, called kernels, such that k is a (possibly infinite) sum $k = \sum_{ij\ell} k_{ij\ell}$, with each $k_{ij\ell} \in C^\infty_c(U_i \times_{\gamma_{ij\ell}} U_j, E \boxtimes E^*)$. For each k, we require that there is a bound on the leafwise length of its $\gamma_{ij\ell}$. We further require that for each k, each of its derivatives in the local coordinates given by the good cover is bounded, with the bound possibly depending on the particular derivative.

The proof of Lemma 2.3 of [5] shows that this is indeed an algebra. Using the K-theory of such a foliated analog of the Roe algebra as a receptacle for indices of leafwise Dirac operators on open foliated manifolds is not a completely new idea. See for instance [32].

Each $k \in C^\infty_u(E \boxtimes E^*)$ defines a $\mathcal{G}$-invariant leafwise smoothing operator on $C^\infty_c(E)$ in the sense of [13], which is transversely smooth, and it has finite propagation due to the limit on the leafwise lengths of its $\gamma_{ij\ell}$. See [30] for the definition of bounded geometry smoothing operators, as well as [28] for the groupoid version. To see this, use the leafwise distance function $d_x(\gamma, \widehat{\gamma})$ on $\widetilde{L}_x$. This is defined as the infimum over the leafwise length $\mathfrak{l}(\gamma\widehat{\gamma}^{-1})$ of all paths in the class of $\gamma\widehat{\gamma}^{-1} \in \mathcal{G}$. For any bounded geometry foliation with Hausdorff holonomy groupoid, the sets $U_i \times_{\gamma_{ij\ell}} U_j$ have the property that there is a universal constant (namely the bound C on the diameters of all the plaques in all the $U_i \times_{\gamma_{ij\ell}} U_j$), so that for all $\gamma \in U_i \times_{\gamma_{ij\ell}} U_j$, we have $\mathfrak{l}(\gamma) \leq \mathfrak{l}(\gamma_{ij\ell}) + 2C$. Next, suppose that $k_{ij\ell} \in C^\infty_c(U_i \times_{\gamma_{ij\ell}} U_j, E \boxtimes E^*)$, and $\sigma \in C^\infty_c(E)$. Then,

$$k_{ij\ell}(\sigma)(\gamma) \;=\; \int\limits_{\widetilde{L}_{s(\gamma)}} k_{ij\ell}(\gamma\widehat{\gamma}^{-1})\sigma(\widehat{\gamma})d\widehat{\gamma}.$$

Now, $k_{ij\ell}(\gamma\widehat{\gamma}^{-1}) = 0$ unless $\gamma\widehat{\gamma}^{-1} \in U_i \times_{\gamma_{ij\ell}} U_j$, that is only if $\mathfrak{l}(\gamma\widehat{\gamma}^{-1}) = d_{s(\gamma)}(\gamma,\widehat{\gamma}) \leq \mathfrak{l}(\gamma_{ij\ell}) + 2C$, the very definition of finite propagation. The restrictions imposed on each $k_{ij\ell}$ imply that each U_i and each U_j appears at most a bounded number of times, so the sum converges locally uniformly, in particular pointwise. These restrictions on k insure that it also has bounded propagation.

Denote by D_F a generalized leafwise Dirac operator for the even dimensional foliation F. It is defined as follows. Let E_M be a complex vector bundle over M with Hermitian metric and connection, which is of bounded geometry. Assume that the tangent bundle TF is spin with a fixed spin structure. Because F is even dimensional, the bundle of spinors along its leaves, denoted $\mathcal{S}_F$ splits as $\mathcal{S}_F = \mathcal{S}_F^+ \oplus \mathcal{S}_F^-$. Denote by ∇^F the Levi-Civita connection on each leaf L of F. ∇^F induces a connection ∇^F on $\mathcal{S}_F|L$, and we denote by $\nabla^{F,E}$ the tensor product connection on $\mathcal{S}_F \otimes E_M|L$. These data determine a smooth family $D_F = \{D_L\}$ of leafwise Dirac operators, where D_L acts on sections of $\mathcal{S}_F \otimes E_M|L$ as follows. Let $X_1, \ldots, X_p$ be a local oriented orthonormal basis of TL, and set

$$D_L \;=\; \sum_{i=1}^{p} \rho(X_i)\nabla^{F,E}_{X_i}$$

where $\rho(X_i)$ is the Clifford action of X_i on the bundle $\mathcal{S}_F \otimes E_M|L$. Then D_L does not depend on the choice of the X_i, and it is an odd operator for the $\mathbb{Z}_2$ grading of $\mathcal{S}_F \otimes E_M = (\mathcal{S}_F^+ \otimes E_M) \oplus (\mathcal{S}_F^- \otimes E_M)$. Thus $D_F : C_c^\infty(\mathcal{S}_F^\pm \otimes E_M) \to C_c^\infty(\mathcal{S}_F^\mp \otimes E_M)$, and $D_F^2 : C_c^\infty(\mathcal{S}_F^\pm \otimes E_M) \to C_c^\infty(\mathcal{S}_F^\pm \otimes E_M)$. For more on the generalized Dirac operators that we are using here, see [27].

Given a leafwise operator A on $\mathcal{S} \otimes E \otimes \wedge\nu_s^*$, denote its leafwise Schwartz kernel by k_A. Then, depending on the context and under appropriate assumptions on k_A, the Haefliger traces, $\mathrm{Tr}(A)$ and $\mathfrak{Tr}(A)$, of A are defined to be,

$$\mathrm{Tr}(A) = \int\limits_F \mathrm{tr}(k_A(\overline{x},\overline{x}))dx_F \in \mathcal{A}_c^*(M/F) \quad \text{and}$$

$$\mathfrak{Tr}(A) = \left[\int\limits_F \mathrm{tr}(k_A(\overline{x},\overline{x}))dx_F\right] \in H_c^*(M/F),$$

where dx_F is the leafwise volume form associated with the fixed orientation of the foliation F. The element $\overline{x} \in \widetilde{L}_x$ is the class of the constant path at $x \in L \subset M$. See again for instance [4] for more details on these constructions.

Now suppose that we have the situation in Section 4 of the companion paper [8]. That is, we have:

- foliated manifolds (M, F) and (M', F');
- Clifford bundles $E_M \to M$ and $E_{M'} \to M'$, with Clifford compatible Hermitian connections;
- leafwise Dirac operators D_F and $D_{F'}$;
- compact subspaces $\mathcal{K}_M = M \smallsetminus V_M$ and $\mathcal{K}'_{M'} = M' \smallsetminus V'_{M'}$;
- an isometry $\varphi : V_M \to V'_{M'}$ with $\varphi^{-1}(F') = F$;
- an isomorphism $\phi : E_M|_{V_M} \to E'_{M'}|_{V'_{M'}}$, covering φ, with $\phi^*(\nabla^{F',E'}|_{V'_{M'}}) = \nabla^{F,E}|_{V_M}$.

The pair $\Phi = (\phi, \varphi)$ is thus a bundle morphism from $E_M|V_M$ to $E'_{M'}|V'_{M'}$. The well defined (since they are differential operators) restrictions of D_F and $D_{F'}$ to the sections over V_M and $V'_{M'}$ agree through Φ, i.e.

$$(\Phi^{-1})^* \circ D_F \circ \Phi^* |_{V'_{M'}} = D_{F'}|_{V'_{M'}}.$$

Such operators are called Φ compatible. Without loss of generality, we may assume that $\mathcal{K}_M$ and $\mathcal{K}'_{M'}$ are the closures of open subsets of M and M' respectively.

Recall the following material from [8]. Denote by $g : M \to [0, \infty)$ and $g' : M' \to [0, \infty)$ compatible smooth approximations to the distance functions $\mathfrak{d}_M(\mathcal{K}_M, x)$ and $\mathfrak{d}_{M'}(\mathcal{K}'_{M'}, x')$, where $\mathfrak{d}_M$ and $\mathfrak{d}_{M'}$ are the distance functions on M and M'. So we assume that g and g' are 0 on $\mathcal{K}_M$ and $\mathcal{K}'_{M'}$ respectively and they satisfy $g' \circ \varphi = g$. Hence, for $s \geq 0$, the open submanifolds $M(s) = \{g > s\}$ and $M'(s) = \{g' > s\}$ agree through φ, that is $\varphi(M(s)) = M'(s)$ and $g|_{M(s)} = g' \circ \varphi|_{M(s)}$. For $s \geq 0$ denote by T_s the set

$$T_s \;=\; \{T_i \subset T \mid T_i \cap M(s) \neq \emptyset\},$$

and similarly for T'_s.

Suppose that $(\zeta, \zeta') \in \mathcal{W} \times \mathcal{W}' \subset \mathcal{A}^*_c(T) \times \mathcal{A}^*_c(T')$, with $\zeta = \sum_{(\alpha,\gamma)} \alpha - h^*_\gamma \alpha$ and $\zeta' = \sum_{(\alpha',\gamma')} \alpha' - h^*_{\gamma'} \alpha'$. For simplicity, we have dropped the subscripts. The vector subspace $\mathcal{W} \times_\varphi \mathcal{W}' \subset \mathcal{W} \times \mathcal{W}'$ consists of elements (ζ, ζ') which are φ compatible. This means that all but a finite number of the (α, γ) and (α', γ') are paired, that is

$$\alpha \;=\; \varphi^*(\alpha') \quad \text{and} \quad \gamma' \;=\; \varphi \circ \gamma, \quad \text{so} \quad \alpha - h^*_\gamma \alpha \;=\; \varphi^*(\alpha' - h^*_{\gamma'} \alpha').$$

Definition 2.2. Given $\beta \in \mathcal{A}^*_c(T)$ and $\beta' \in \mathcal{A}^*_c(T')$, the pair (β, β') is φ-compatible if there exists $s \geq 0$ so that $\beta = \varphi^*(\beta')$ on T_s. Set

$$\mathcal{A}^*_c(M/F, M'/F'; \varphi) \;=\; \{(\beta, \beta') \in \mathcal{A}^*_c(T) \times \mathcal{A}^*_c(T') \mid (\beta, \beta') \text{ is } \varphi \text{ compatible}\}/(\overline{\mathcal{W} \times_\varphi \mathcal{W}'}).$$

The de Rham differentials on $\mathcal{A}^*_c(T)$ and $\mathcal{A}^*_c(T')$ yield a well defined relative Haefliger complex, whose homology spaces are denoted

$$H^*_c(M/F, M'/F'; \varphi) = \oplus_{0 \leq k \leq q} H^k_c(M/F, M'/F'; \varphi),$$

and there are well defined graded maps,

$$\pi : H_c^*(M/F, M'/F'; \varphi) \to H_c^*(M/F) \text{ and } \pi' : H_c^*(M/F, M'/F'; \varphi) \to H_c^*(M'/F').$$

which are induced by the projections

$$\mathcal{A}_c^*(M/F, M'/F'; \varphi) \to \mathcal{A}_c^*(M/F) \text{ and } \mathcal{A}_c^*(M/F, M'/F'; \varphi) \to \mathcal{A}_c^*(M'/F').$$

Definition 2.3. Suppose $(\xi, \xi') \in \mathcal{A}_c^*(M/F, M'/F'; \varphi)$, and let C and C' be closed (bounded) φ compatible holonomy invariant Haefliger currents. Set

$$\langle (\xi, \xi'), (C, C') \rangle \;=\; \lim_{s \to \infty} \left(C(\xi|_{T \smallsetminus T_s}) - C'(\xi'|_{T' \smallsetminus T'_s}) \right).$$

This is well defined because any representative in (ξ, ξ') is φ compatible, so the right hand side is eventually constant. In addition, every $(\zeta, \zeta') \in \mathcal{W} \times_\varphi \mathcal{W}'$ is φ compatible, so satisfies

$$\lim_{s \to \infty} \left(C(\zeta|_{T \smallsetminus T_s}) - C'(\zeta'|_{T' \smallsetminus T'_s}) \right) \;=\; 0.$$

To see this, recall that there is a global bound on the leafwise length of the γ and γ' in ζ and ζ'. This, and the fact that there are only finitely many unpaired (α, γ) and (α', γ'), insures that for large s, every unpaired (α, γ) will have both Dom_γ and $\mathrm{Ran}_\gamma \subset T \smallsetminus T_s$, so $C(\alpha - h_\gamma^* \alpha)$ will be zero, and similarly for every unpaired (α', γ'). Those (α, γ) and (α', γ') which are paired and appear in the integration, will have Dom_γ and/or $\mathrm{Ran}_\gamma \subset T \smallsetminus T_s$ with corresponding $\mathrm{Dom}_{\gamma'}$ and/or $\mathrm{Ran}_{\gamma'} \subset T' \smallsetminus T'_s$. In both cases, their integrals will cancel.

Remark 2.4. Examples of such currents include the following.

1. Invariant transverse measures Λ and Λ' on T and T' which are φ compatible as in [8].
2. Suppose $\omega \in C^\infty(\wedge^* \nu^*)$ and $\omega' \in C^\infty(\wedge^* \nu'^*)$ are closed holonomy invariant forms on M and M' which are φ compatible. They determine φ compatible closed holonomy invariant currents, also denoted ω_T and $\omega'_{T'}$. In particular,

$$\langle (\xi, \xi'), (\omega_T, \omega'_{T'}) \rangle \;=\; \lim_{s \to \infty} \left(\int_{T \smallsetminus T_s} \xi \wedge \omega_T - \int_{T' \smallsetminus T'_s} \xi' \wedge \omega'_{T'} \right).$$

   Here $\omega_T = \omega\,|_T$, which is well defined and is holonomy invariant, as is $\omega'_{T'}$.
   For Riemannian foliations, examples of this type abound. In particular, the characteristic forms of holonomy invariant bundles which agree at infinity, for example $\wedge^j \nu^* \otimes (\otimes^\ell \nu)$, and $\wedge^j \nu'^* \otimes (\otimes^\ell \nu')$.

For definiteness, we will generally use this example in the sequel, but all the statements obviously remain valid with more general holonomy invariant currents.

In this paper, we will have a number of different pairings, which will be uniformly indicated by the notation $\langle \cdot, \cdot \rangle$. The notation should make clear where the objects live. For example, we have

$$\langle \left[\int_F \mathrm{AS}(D_F), \int_{F'} AS(D'_{F'}) \right], [\omega_T, \omega'_{T'}] \rangle = \int_T \left(\int_F \mathrm{AS}(D_F) \right) \wedge \omega_T - \int_{T'} \left(\int_{F'} AS(D'_{F'}) \right) \wedge \omega'_{T'},$$

and

$$\langle (\mathrm{ch}(P_0), \mathrm{ch}(P_0)), (\omega_T, \omega'_{T'}) \rangle = \int_T \mathrm{ch}(P_0) \wedge \omega_T - \int_{T'} \mathrm{ch}(P'_0) \wedge \omega'_{T'}.$$

In the first case, the terms in the pairing live in relative Haefliger cohomology. In the second, the terms are pairs of bounded Haefliger forms, and the second pair happen to agree near infinity.

3. Chern characters in haefliger cohomology

We recall in this section the main steps in the construction of the Chern character in Haefliger cohomology and explain how they immediately extend to the case of a pair of foliations which are compatible near infinity. In this latter case, our Chern character takes values in a relative version of Haefliger cohomology that we introduce below.

In [8] we worked on M, while in [23,25,4,5], we worked on $\mathcal{G}$, which we will also do here, but our basic data will be taken from the ambient manifolds. The results in [8] extend readily to $\mathcal{G}$ with the only change being that the spectral projections used on $\mathcal{G}$ are for the operator lifted to F_s. This represents another extension, in the spirit of Connes' extensions in [13,14], of the classical Atiyah L^2 covering index theorem, [1]. In addition, as will be explained below, the results in the above cited papers where M was assumed to be compact still hold provided both M and F are of bounded geometry and we use our definition of the Haefliger cohomology.

All the data in the previous section may be lifted to $(\mathcal{G}, F_s)$ using the map $r : \mathcal{G} \to M$. The notation we will use is obtained from that above by:

$$E_M \to E; \quad \mathcal{S}_F \to \mathcal{S}; \quad \nabla^{F,E} \to \nabla; \quad L \to \widetilde{L}_x; \quad D_F \to D; \quad D_L \to D_x.$$

Thus the smooth $\mathcal{G}$ invariant family $D = \{D_x\}$ of leafwise Dirac operators acting on sections of $\mathcal{S} \otimes E|\widetilde{L}_x$ is given as follows. Let $X_1, \ldots, X_p$ be a local oriented orthonormal basis of $T\widetilde{L}_x$. Then,

$$D_x = \sum_{i=1}^{p} \rho(X_i)\nabla_{X_i} \ : \ C_c^\infty(\mathcal{G}_x, \mathcal{S}^\pm \otimes E) \to C_c^\infty(\mathcal{G}_x, \mathcal{S}^\mp \otimes E) \text{ and}$$
$$D_x^2 : C_c^\infty(\mathcal{G}_x, \mathcal{S}^\pm \otimes E) \to C_c^\infty(\mathcal{G}_x, \mathcal{S}^\pm \otimes E).$$

Denote by $\wedge\nu_s^*$, the exterior powers of the dual normal bundle ν_s^* of $F_s = r^*F$, which we identify with $s^*(T^*M) = s^*(TF^*) \oplus s^*(\nu^*)$ so that each $C_c^\infty(\mathcal{S} \otimes E \otimes \wedge\nu_s^*)$ is an $\Omega^*(M)$-module. We extend D to an $\Omega^*(M)$-equivariant operator

$$D : C_c^\infty(\mathcal{S} \otimes E \otimes \wedge\nu_s^*) \longrightarrow C_c^\infty(\mathcal{S} \otimes E \otimes \wedge\nu_s^*),$$

by using the leafwise flat connection on $\wedge\nu_s^*$ determined by the pull-back of the Levi-Civiti connection on T^*M.

In [5], we used the traces Tr and $\mathfrak{Tr}$ to define Connes-Chern characters in $H_c^*(M/F)$ for operators on $C_c^\infty(\mathcal{S} \otimes E)$. For the leafwise spectral projection P_0 onto the kernel of D^2, when this latter is smooth, this is denoted,

$$\mathrm{ch}(P_0) \ \in \ \mathrm{H}_c^*(M/F).$$

We also proved that if M is compact and $\mathrm{Ind}_a(D)$ is Connes' K-theory index class defined in terms of a parametrix for D, then under the usual regularity assumption, $\mathrm{ch}(P_0) = \mathrm{ch}(\mathrm{Ind}_a(D))$. We now extend these notions to our situation.

We now return to our compatible foliations (M, F) and (M', F') and their holonomy groupoids $\mathcal{G}$ and $\mathcal{G}'$. First, we lift the compatibility data Φ to $\mathcal{G}$ and denote again the corresponding data by Φ, which gives an equivalence off (the generally non-compact subsets) $K = r^{-1}(K_M)$ and $K' = (r')^{-1}(K_{M'})$, that is on the subsets $V = r^{-1}(V_M)$ and $V' = (r')^{-1}(V_{M'})$. In [5], we defined an algebra of super-exponentially decaying $\mathcal{G}$-operators on $C_c^\infty(\mathcal{S} \otimes E \otimes \wedge\nu_s^*)$. Here we need a stronger condition on our operators, namely that they have finite propagation. This is provided by using operators from the algebra $C_u^\infty((\mathcal{S}\otimes E\otimes\wedge\nu_s^*)\boxtimes(\mathcal{S}\otimes E\otimes\wedge\nu_s^*)^*)$, which we denote simply as $C_u^\infty(F_s)$. Any $A = (A_x)_{x\in M} \in C_u^\infty(F_s)$ defines a leafwise (smoothing) $\mathcal{G}$-operator on $C_c^\infty(\mathcal{S}\otimes E\otimes\wedge\nu_s^*)$ which has uniform finite propagation, and its Schwartz kernel is smooth in all variables, with all derivatives being globally bounded, the bounds possibly depending on the derivatives.

Using the algebra $C_u^\infty(F_s)$, we have a K-theory index class represented by idempotents constructed from a parametrix, and this K-index does not depend on the parametrix, so its Connes-Chern character is also independent of the parametrix. For a more detailed discussion of the following construction, see [16], p. 353. In particular, as D is an odd super operator, we may write $D = \begin{bmatrix} 0 & D^- \\ D^+ & 0 \end{bmatrix}$. Suppose that Q_t is a smooth (in t)

family of leafwise parametrics for D. That is, each Q_t is an odd operator which is smooth in all variables, and which has finite propagation remainders, namely the even operators

$$S_t \;=\; \mathrm{I}_{\mathcal{S}^+\otimes E} - Q_t{}^- D^+ \quad \text{and} \quad R_t \;=\; \mathrm{I}_{\mathcal{S}^-\otimes E} - D^+ Q_t{}^-.$$

For $t > 0$, set, as in [5],

$$A_t = \begin{bmatrix} S_t^2 & Q_t^-(R_t + R_t^2) \\ R_t D^+ & -R_t^2 \end{bmatrix}.$$

Then A_t has finite propagation, is smooth in all variables, and is a bounded leafwise smoothing operator, that is, $A_t \in C_u^\infty(F_s)$. Set $\pi_- = \mathrm{diag}(0, \mathrm{I}_{\mathcal{S}^-\otimes E})$, and $\pi_+ = \mathrm{diag}(\mathrm{I}_{\mathcal{S}^+\otimes E}, 0)$. Then $A_t + \pi_-$ is an idempotent as is π_-. Set

$$\mathrm{Ind}_a(D) \;=\; [A_t + \pi_-] - [\pi_-] \in \mathrm{K}_0(C_u^\infty(F_s)).$$

Since $A_t + \pi_-$ is a smooth family of idempotents, it follows from results of [4] that $\mathrm{Ind}_a(D)$ is independent of t. Since any two parametrics can be joined in a smooth family, it follows immediately that $\mathrm{Ind}_a(D)$ does not depend on the parametrix.

For details of the following, see [5], Section 3, where we define the quasi-connection,

$$C^\infty(\mathcal{S}\otimes E\otimes \wedge\nu_s^*) \xrightarrow{\nabla^\nu} C^\infty(\mathcal{S}\otimes E\otimes \wedge\nu_s^*).$$

Given an operator A on $\mathcal{S}\otimes E\otimes\wedge\nu_s^*$, denote by

$$\partial_\nu : \mathrm{End}(C^\infty(\mathcal{S}\otimes E\otimes\wedge\nu_s^*)) \to \mathrm{End}(C^\infty(\mathcal{S}\otimes E\otimes\wedge\nu_s^*))$$

the linear operator given by the graded commutator

$$\partial_\nu(A) = [\nabla^\nu, A].$$

Set $\theta = (\nabla^\nu)^2$, which is a leafwise differential operator with coefficients in $\wedge\nu_s^*$. Since ∂_ν^2 is not necessarily zero, in [5] we used Connes' X-trick, see [17], Section III.3, Lemma 9, to construct a new differential operator δ out of ∂_ν and θ, whose square is zero. Note carefully that δA is nilpotent since it always contains a coefficient from $\wedge\nu_s^{*\geq 1}$.

Corollary 3.7 of [5] states,

Proposition 3.1. *The Haefliger form* $\mathrm{Tr}\Big(A_t \exp\Big[\frac{-(\delta A_t)^2}{2i\pi}\Big]\Big)$ *is closed, and the Haefliger class*

$\mathfrak{Tr}\Big(A_t \exp\Big[\frac{-(\delta A_t)^2}{2i\pi}\Big]\Big)$ *is independent of* t.

Definition 3.2. The Connes-Chern character of $\mathrm{Ind}_a(D)$ is,

$$\mathrm{ch}(\mathrm{Ind}_a(D)) = \mathfrak{Tr}\Big(A_t \exp\left[\frac{-(\delta A_t)^2}{2i\pi}\right]\Big) \ \in \ H_c^*(M/F).$$

We have the same constructions for D'. In Section 5, we construct families of parametrics Q_t and Q'_t directly from D and D' in such a way that their remainders are Φ compatible, so also are A_t and A'_t.

For pairs (A, A') of operators from $C_u^\infty(F_s) \times C_u^\infty(F'_s)$ which are Φ-compatible, there is also an algebra $C_u^\infty(F_s, F'_s; \Phi)$, and the previous construction of the analytic index class extends immediately to yield the relative analytic index class

$$\mathrm{Ind}_a(D, D') \ = \ [(A_t + \pi_-, A'_t + \pi'_-] - [(\pi_-, \pi'_-)] \in \mathrm{K}_0(C_u^\infty(F_s, F'_s; \Phi)).$$

The Connes-Chern character then extends to the relative case

$$\mathrm{ch} : \mathrm{K}_0(C_u^\infty(F_s, F'_s; \Phi)) \longrightarrow H_c^*(M/F, M'/F'; \varphi),$$

with the obvious definition (see [5], Theorem 3.2 for the notation below and more precise details),

$$\mathrm{ch}([\tilde{e}, \tilde{e}']) \ = \ \left[\mathrm{Tr}\left(e \exp\left(\frac{-(\delta e)^2}{2i\pi}\right)\right), \mathrm{Tr}\left(e' \exp\left(\frac{-(\delta e')^2}{2i\pi}\right)\right)\right] \ \in \ H_c^*(M/F, M'/F'; \varphi).$$

Definition 3.3. Suppose the parametrics Q_t and Q'_t have Φ compatible remainders, so with Φ compatible operators A_t and A'_t. Then the relative Connes-Chern character of $\mathrm{Ind}_a(D, D')$ is given by

$$\mathrm{ch}(\mathrm{Ind}_a(D, D')) =$$
$$\left[\mathrm{Tr}\left(A_t \exp\left(\frac{-(\delta A_t)^2}{2i\pi}\right)\right), \mathrm{Tr}\left(A'_t \exp\left(\frac{-(\delta A'_t)^2}{2i\pi}\right)\right)\right] \ \in \ H_c^*(M/F, M'/F'; \varphi).$$

The class $\mathrm{ch}(\mathrm{Ind}_a(D, D'))$ is clearly well defined due to its independence of the Φ-compatible pair of finite propagation parametrics. This is proved below, see Theorem 5.5, where we also point out that it is independent of the parameter t.

4. Four theorems

Our first main theorem is the following extension of a classical Atiyah-Singer Index Theorem. This theorem is purely local and, as in [8], requires bounded geometry.

Denote by $[AS(D_F)]$ the Atiyah-Singer characteristic class for D_F, and similarly for $D'_{F'}$. Note that for large s, the differential forms satisfy $AS(D_F) = \varphi^*(AS(D'_{F'}))$ on M_s, so

$$\left(\int_F \mathrm{AS}(D_F), \int_{F'} AS(D'_{F'})) \right) \ \in \ \mathcal{A}_c^*(M/F, M'/F'; \varphi).$$

Definition 4.1. The relative topological index of (D, D') is,

$$\mathrm{Ind}_t(D, D') \ = \ \left[\int_F \mathrm{AS}(D_F), \int_{F'} AS(D'_{F'})) \right] \ \in \ H_c^*(M/F, M'/F'; \varphi).$$

Theorem 4.2. *[The Higher Relative Index Theorem] Suppose that (M, F), (M', F'), D and D' are as in Section 2. In particular, F and F' need not be Riemannian. Then,*

$$\mathrm{ch}(\mathrm{Ind}_a(D, D')) = \mathrm{Ind}_t(D, D') \ \in H_c^* \ (M/F, M'/F'; \varphi)$$

In particular, for any closed φ-compatible pair (C, C') of holonomy invariant closed Haefliger currents, the following scalar formula holds

$$\langle \mathrm{ch}(\mathrm{Ind}_a(D, D')), [C, C'] \rangle = \lim_{s \to +\infty} \left(\langle \int_F \mathrm{AS}(D_F)|_{T \smallsetminus T_s}, C \rangle - \langle \int_{F'} \mathrm{AS}(D'_{F'})|_{T' \smallsetminus T'_s}, C' \rangle \right).$$

.

Denote by $P_{(0,\epsilon)}$ the spectral projection for D^2 for the interval $(0, \epsilon)$. The Novikov-Shubin invariants $NS(D)$ of D are greater than $k \geq 0$ provided that there is $\tau > k$ so that

$$\mathrm{Tr}(P_{(0,\epsilon)}) \text{ is } \mathcal{O}(\epsilon^\tau) \text{ as } \epsilon \to 0.$$

A Haefliger form Ψ depending on ϵ is $\mathcal{O}(\epsilon^\tau)$ as $\epsilon \to 0$ means that there is a representative $\psi \in \Psi$ defined on a transversal T, and a constant $C > 0$, so that the function on T, $\|\psi\|_T \leq C\epsilon^\tau$ as $\epsilon \to 0$. Here $\| \quad \|_T$ is the pointwise norm on forms on the transversal T induced from the metric on M.

Recall that P_0 is the spectral projection onto the kernel of D^2. In general the leafwise operators $P_{(0,\epsilon)}$ and P_0 are not transversely smooth (although they are always leafwise smooth), so that, in general, their Haefliger traces in $\mathcal{A}_c^*(M/F)$ are not defined. When P_0 is transversely smooth, see [5], Definition 3.8,

$$\mathrm{ch}(P_0) \ = \ \mathfrak{Tr}\Big(\pi_\pm P_0 \exp\left(\frac{-(\delta(\pi_\pm P_0))^2}{2i\pi} \right) \Big) \ \in \ H_c^*(M/F),$$

and similarly for P'_0. Here $\pi_\pm$ is the grading operator

$$\pi_\pm \ = \ \mathrm{diag}(\mathrm{I}_{\mathcal{S}^+ \otimes E}, -\mathrm{I}_{\mathcal{S}^- \otimes E}).$$

When $P_{(0,\epsilon)}$ is transversely smooth,

$$\mathrm{ch}(P_{(0,\epsilon)}) \;=\; \mathfrak{Tr}\Big(\pi_\pm P_{(0,\epsilon)} \exp\left(\frac{-(\delta(\pi_\pm P_{(0,\epsilon)}))^2}{2i\pi}\right)\Big) \;\in\; H_c^*(M/F),$$

and similarly for $P'_{(0,\epsilon)}$ For simplicity of notation, we will uniformly suppress the constant $2i\pi$ in what follows. As the closed Haefliger differential forms $\mathrm{Tr}(\pi_\pm P_0 \exp\left(-(\delta(\pi_\pm P_0))^2\right))$ and $\mathrm{Tr}(\pi'_\pm P'_0 \exp\left(-(\delta(\pi'_\pm P'_0))^2\right))$ are not φ compatible in general, we proceed as follows.

The component of $\mathrm{ch}(\mathrm{Ind}_a(D, D'))$ in $H_c^{2k}(M/F, M'/F'; \varphi)$ is denoted $\mathrm{ch}^k(\mathrm{Ind}_a(D, D'))$, and the part of $\mathrm{ch}(\mathrm{Ind}_a(P_0)$ in $H_c^{2k}(M/F)$ is denoted $\mathrm{ch}^k(P_0))$, and similarly for P'_0.

The following theorem generalizes the main result of [5] to bounded geometry foliations.

Theorem 4.3. *[Riemannian Foliation Relative Index Bundle Theorem] Fix $0 \leq \ell \leq q/2$, where q is the codimension of F and F'. Assume that:*

- *the foliations F and F' are Riemannian;*
- *the leafwise operators P_0, P'_0, $P_{(0,\epsilon)}$ and $P'_{(0,\epsilon)}$ (for ϵ sufficiently small) are transversely smooth;*
- *$NS(D)$ and $NS(D')$ are greater than ℓ.*

Then, for $0 \leq k \leq \ell$, we have in $H_c^{2k}(M/F) \times H_c^{2k}(M'/F')$

$$(\pi \times \pi')\,\mathrm{ch}^k(\mathrm{Ind}_a(D, D')) \;=\; (\mathrm{ch}^k(\mathrm{Ind}_a(D)), \mathrm{ch}^k(\mathrm{Ind}_a(D'))) \;=\; (\mathrm{ch}^k(P_0), \mathrm{ch}^k(P'_0)).$$

Remarks 4.4.

1. If the foliations F and F' are not Riemannian then we can still prove this equality but under the stronger assumption that $NS(D)$ and $NS(D')$ be greater than $3q$, see [25,9].
2. The examples in [6] show that the conditions on the Novikov-Shubin invariants are the best possible.
3. Note that if there are uniform gaps in the spectrums at 0, that is there is $\epsilon > 0$ so $P_{(0,\epsilon)} = P'_{(0,\epsilon)} = 0$, then $P_{(0,\epsilon)}$ and $P'_{(0,\epsilon)}$ are transversely smooth and the Novikov-Shubin invariants are infinite. For top dimensional foliations, i.e. $TF = TM$, these special cases were studied for instance in [33,18].

Combining Theorem 4.2 and Theorem 4.3, we immediately deduce the following important corollary.

Theorem 4.5. *Under the assumptions of Theorem 4.3, assume furthermore that $P_0 = P'_0 = 0$, then*

$$\left(\int_F \mathrm{AS}(D_F), \int_{F'} AS(D'_{F'}) \right) \; = \; (0,0) \quad \textit{in } H_c^*(M/F) \times H_c^*(M'/F').$$

So the vanishing conclusion of the previous theorem holds in particular when there exists $\epsilon > 0$ such that $P_{[0,\epsilon)} = 0$ and $P'_{[0,\epsilon)} = 0$.

Denote by $\omega \in C^\infty(\wedge^* \nu^*)$ and $\omega' \in C^\infty(\wedge^* \nu'^*)$ closed bounded holonomy invariant differential forms on M and M' which are φ compatible. For simplicity, we will assume that ω and ω' are φ compatible on V_M and $V'_{M'}$. These determine φ compatible closed bounded Haefliger forms on T, denoted ω_T and $\omega'_{T'}$. Recall that dx is the global volume form on M.

Theorem 4.6. *[Higher Relative Index Pairing Theorem] In addition to the assumptions in Theorem 4.3, assume the following:*

- *for ϵ sufficiently small, $P_{[0,\epsilon)}$ satisfies $\int_M \mathrm{tr}(P_{[0,\epsilon)})dx \; < \; \infty$, and similarly for $P'_{[0,\epsilon)}$;*
- *M, and so also M', has sub-exponential growth.*

Then, for any homogeneous $\omega \in C^\infty(\wedge^{q-2k} \nu^*)$ and $\omega' \in C^\infty(\wedge^{q-2k} \nu'^*)$ as above, $(0 \leq k \leq \ell)$,

$$\int_T \mathrm{ch}(P_0) \wedge \omega_T \textit{ and } \int_{T'} \mathrm{ch}(P'_0) \wedge \omega'_{T'} \textit{ are well defined complex numbers,}$$

and

$$\int_T \mathrm{ch}(P_0) \wedge \omega_T \; - \int_{T'} \mathrm{ch}(P'_0) \wedge \omega'_{T'} \; = \; \langle \left[\int_F \mathrm{AS}(D_F), \int_{F'} AS(D'_{F'}) \right], [\omega_T, \omega'_{T'}] \rangle.$$

Remarks 4.7.

1. Since the pair of Connes-Chern characters of P_0 and P'_0 is usually not φ-compatible, the previous theorem is totally new and we cannot deduce it from any absolute version of the index bundle theorem. This is compatible with the classical relative index theorem.
2. The theorem also holds for appropriate closed φ compatible closed holonomy invariant currents, but this more general statement will not be needed for our applications.
3. We shall see in Section 6 that the finite integral assumptions are satisfied when the zero-th order operator $\mathcal{R}_F^E$ defined there in the Bochner formula is strictly positive near infinity. As $\mathcal{R}_F^E$ is locally defined, this means that $\mathcal{R}_{F'}^{E'}$ is also strictly positive near infinity.

4. The growth condition is a technical assumption which simplifies the proof, it can be weakened as explained in Remark 5.7
5. The main theorem in [8] recovers the Gromov-Lawson relative index theorem in full generality for bounded geometry manifolds, which correspond to top-dimensional foliations. Our results here require more conditions to deal with the higher components of the Connes-Chern character, and it only recovers the Gromov-Lawson results for sub-exponential bounded geometry manifolds. Recall that in the top-dimensional case, Gromov-Lawson show that there is $\epsilon > 0$ so that $P_{(0,\epsilon)} = 0$, and $\int_M \mathrm{tr}(P_0)dx < \infty$, so all the other assumptions of Theorems 4.3 and 4.6 are fulfilled.

5. Proofs of the theorems

This section is devoted to the proofs of Theorems 4.2, 4.3, 4.5 and 4.6. The proofs are rather technical and have been split into many intermediate lemmas and propositions. We shall first prove Theorem 4.2 and then later on Theorems 4.3 and 4.5, and eventually we shall end this section by the proof of Theorem 4.6.

Recall the following construction from [8]. Denote the Fourier Transform of a complex valued function g by $\widehat{g}$ and $FT(g)$, and its inverse transform $FT^{-1}(g)$ by $\widetilde{g}$. If h is also a complex function, denote the convolution of g and h by $g \star h$. Set $g_\lambda(z) = g(\lambda z)$, for non-zero $\lambda \in \mathbb{R}^*$. We have the following facts:

$$FT(g_\lambda) = \frac{1}{\lambda} FT(g)_{\frac{1}{\lambda}};\ FT(g \star h) = \sqrt{2\pi} FT(g)FT(h);\ \text{and}$$
$$FT(\widehat{g}) = FT^{-1}(\widehat{g}) = g,\ \text{if } g \text{ is even.}$$

Fix a smooth even non-negative function ψ supported in $[-1, 1]$, which equals 1 on $[-1/4, 1/4]$, is non-increasing on $\mathbb{R}_+$, and whose integral over $\mathbb{R}$ is 1. Note that $FT(\widehat{\psi}) = \psi$ since ψ is even. The family $\frac{1}{\sqrt{t}}\widehat{\psi}_{\frac{1}{\sqrt{t}}}$ is an approximate identity when acting on a Schwartz function f by convolution, since, up to the constant $\sqrt{2\pi}$ which we systematically ignore,

$$\frac{1}{\sqrt{t}}\widehat{\psi}_{\frac{1}{\sqrt{t}}} \star f \ = \ FT^{-1}(FT(\frac{1}{\sqrt{t}}\widehat{\psi}_{\frac{1}{\sqrt{t}}} \star f)) \ = \ FT^{-1}(\psi_{\sqrt{t}}\widehat{f}) \to \widetilde{\widehat{f}} \ = \ f,$$

in the Schwartz topology as $t \to 0$. Denote as usual by $||\cdot||_{r,s}$ the norm of an operator acting from the r Sobolev space to the s Sobolev space. Then more is true.

Lemma 5.1. *Suppose that $\mu : \mathbb{R}_+ \to \mathbb{R}_+$, with $\mu(t) \leq C_p t^p$ or $\mu(t) \geq C_p t^{-p}$ near 0, where $p > 0$ and $C_p > 0$. Then, for any Schwartz function f,*

$$\lim_{t\to 0} \left(\left[\frac{1}{\sqrt{t}}\widehat{\psi}_{\frac{1}{\sqrt{t}}} \star f \right]_{\mu(t)} - \ f_{\mu(t)} \right) = 0$$

in the Schwartz topology.

Thus for all r, s,

$$\lim_{t\to 0} || \left[\frac{1}{\sqrt{t}} \widehat{\psi}_{\frac{1}{\sqrt{t}}} \star f \right]_{\mu(t)} (D) \; - \; f_{\mu(t)}(D) ||_{r,s} \; = \; 0,$$

so the differences of their Schwartz kernels converge uniformly to 0 *pointwise.*

Proof. The last statement follows from standard Sobolev theory given the first. Thus we need only prove that the difference of the Fourier transforms goes to zero in the Schwartz topology. But,

$$FT \left(\left[\frac{1}{\sqrt{t}} \widehat{\psi}_{\frac{1}{\sqrt{t}}} \star f \right]_{\mu(t)} \right) - FT \left[f_{\mu(t)} \right] = \frac{1}{\mu(t)} \widehat{f}_{\frac{1}{\mu(t)}} (\psi_{\sqrt{t}/\mu(t)} - 1).$$

Now, $\psi_{\sqrt{t}/\mu(t)}(z) - 1$ is 0 for $|z| \leq \mu(t)/4\sqrt{t}$ and constant for $|z| \geq \mu(t)/\sqrt{t}$, so all its derivatives are zero on these subsets. In addition, for all non-negative n, there is a constant C_n so that

$$|\frac{\partial^n}{\partial z^n} (\psi_{\sqrt{t}/\mu(t)}(z) - 1)| \; \leq \; C_n (\sqrt{t}/\mu(t))^n.$$

Thus, we have

$$\begin{aligned}
&||z^n \frac{\partial^m}{\partial z^m} \left[\frac{1}{\mu(t)} \widehat{f}_{\frac{1}{\mu(t)}} (\psi_{\sqrt{t}/\mu(t)} - 1) \right] ||_\infty = \\
&\sup_{|z| \geq \mu(t)/(4\sqrt{t})} \left| z^n \frac{\partial^m}{\partial z^m} \left[\frac{1}{\mu(t)} \widehat{f}_{\frac{1}{\mu(t)}} (\psi_{\sqrt{t}/\mu(t)} - 1) \right] \right| \; \leq \\
&\sup_{|z| \geq \mu(t)/(4\sqrt{t})} \left| z^n \sum_{k=0}^{m} C_{m-k} (\sqrt{t}/\mu(t))^{m-k} \frac{\partial^k}{\partial z^k} \left[\frac{1}{\mu(t)} \widehat{f}_{\frac{1}{\mu(t)}} \right] (z) \right| \; = \\
&\sum_{k=0}^{m} C_{m-k} (\sqrt{t}/\mu(t))^{m-k} \mu(t)^{-(k+1)} \sup_{|z| \geq \mu(t)/(4\sqrt{t})} \left| z^n \widehat{f}^{(k)} (z/\mu(t)) \right| \; = \\
&\sum_{k=0}^{m} C_{m-k} (\sqrt{t})^{m-k} \mu(t)^{-(m+1)} \sup_{|z| \geq 1/(4\sqrt{t})} \left| z^n \widehat{f}^{(k)}(z) \right|.
\end{aligned}$$

Since f, so also $\widehat{f}$, is Schwartz, for any non-negative $k \in \mathbb{Z}$, the function $z \mapsto z^n \widehat{f}^{(k)}(z)$ is Schwartz. But for any Schwartz function g, any $N \geq 0$ ($N \leq 0$ is trivial) and any $\eta > 0$, $\lim_{t\to 0} t^{-N} \sup_{|z| \geq \eta/\sqrt{t}} |g(z)| = 0$. Thus, if $\mu(t) \leq C_p t^p$ or $\mu(t) \geq C_p t^{-p}$ near 0, $\lim_{t\to 0} ||z^n \frac{\partial^m}{\partial z^m} \left[\frac{1}{\mu(t)} \widehat{f}_{\frac{1}{\mu(t)}} (\psi_{\sqrt{t}/\mu(t)} - 1) \right] ||_\infty \; = \; 0.$ □

Define the functions $\alpha(t)$ and $\beta(t)$ as follows. Both have domains $(0,1)$, and are smooth. $\alpha(t) = t$ near 0, and $\alpha(t) = 1-t$ near 1, it is increasing on $(0,1/2]$ and symmetric about $t = 1/2$. β is an increasing function, with $\beta = t$ near 0, and $\beta(t) = (1-t)^{-1}$ near 1.

Set $e(z) = e^{-z^2/2}$, and for $t \in (0,1)$, set

$$\chi^t(z) \;=\; \left[\frac{1}{\sqrt{\alpha(t)}}\widehat{\psi}_{\frac{1}{\sqrt{\alpha(t)}}} \star e\right]_{\sqrt{\beta(t)}}(z).$$

Remark 5.2. By Lemma 5.1, we have,

$$\lim_{t\to 0}\left(\chi^t(z) \;-\; e^{-tz^2/2}\right) \;=\; 0 \;=\; \lim_{t\to 1}\left(\chi^t(z) \;-\; e^{-z^2/(2(1-t))}\right),$$

in the Schwartz topology. In addition, the limit as $t \to 0$ of the Schwartz kernel of $\chi^t(D) \;-\; e^{-tD^2/2}$ and the limit as $t \to 1$ of the Schwartz kernel of $\chi^t(D) \;-\; e^{-D^2/(2(1-t))}$ both converge uniformly pointwise to zero.

Lemma 5.3. *$\chi^t(D)$ has finite propagation $\leq \sqrt{\beta(t)/\alpha(t)}$.*

Proof. Since $\widehat{e} = e$, we have that up to a constant,

$$FT\left(\frac{1}{\sqrt{\alpha(t)}}\widehat{\psi}_{\frac{1}{\sqrt{\alpha(t)}}} \star e\right) \;=\; \psi_{\sqrt{\alpha(t)}}e.$$

In fact, up to a constant,

$$\chi^t(D) \;=\; FT^{-1}(\psi_{\sqrt{\alpha(t)}}e)(\sqrt{\beta(t)}D) \;=\; \int_{\mathbb{R}} \psi(\sqrt{\alpha(t)}\xi)e(\xi)\cos(\xi\sqrt{\beta(t)}D)\,d\xi,$$

since $\psi_{\sqrt{\alpha(t)}}e$ is even. Setting $\eta = \sqrt{\alpha(t)}\xi$ gives,

$$\chi^t(D) = \frac{1}{\sqrt{\alpha(t)}}\int_{|\eta|\leq 1} \psi(\eta)e(\eta/\sqrt{\alpha(t)})\cos(\eta\sqrt{\beta(t)/\alpha(t)}D)d\eta.$$

The operator $\cos(\eta\sqrt{\beta(t)/\alpha(t)}D)$ has propagation $\leq |\eta\sqrt{\beta(t)/\alpha(t)}|$, see [12,29]. Thus $\chi^t(D)$ has finite propagation $\leq \sqrt{\beta(t)/\alpha(t)}$, which near 0 is ≤ 1, while near 1 it is $\leq (1-t)^{-1}$, so may go to infinity as $t \to 1$. □

As D is an odd super operator, we may write

$$D = \begin{bmatrix} 0 & D^- \\ D^+ & 0 \end{bmatrix}, \text{ and we set}$$

$$Q_t = \Big(\frac{1-\chi^t(0)^{-1}\chi^t(z)}{z}\Big)(D) \;=\; \Big(\frac{1-\chi^t(0)^{-1}\chi^t(z)}{z^2}z\Big)(D).$$

We claim that Q_t is a smooth family of leafwise parametrics for D with finite propagation Φ compatible remainders, namely the even operators

$$S_t \;=\; \mathrm{I}_{\mathcal{S}^+\otimes E} - Q_t{}^- D^+ \quad \text{and} \quad R_t \;=\; \mathrm{I}_{\mathcal{S}^-\otimes E} - D^+ Q_t^-.$$

There are similar relations for D^-.

The main step in the proof of Theorem 4.2 is the following expected independent result.

Proposition 5.4. *For $0 < t < 1$, set, as in [8],*

$$A_t = \begin{bmatrix} S_t^2 & Q_t^-(R_t + R_t^2) \\ R_t D^+ & -R_t^2 \end{bmatrix} = \begin{bmatrix} S_t^2 & S_t Q_t^-(1+R_t) \\ R_t D^+ & -R_t^2 \end{bmatrix},$$

a form which is more useful here. Then A_t and δA_t, so also $(\delta A_t)^2$, have finite propagations which are bounded by multiples of $\sqrt{\beta(t)/\alpha(t)}$, are smooth in all variables, and are bounded leafwise smoothing operators.

Proof. We deal with A_t first. Note that $S_t = \chi^t(0)^{-1}\chi^t(D)$ acting on $\mathcal{S}^+ \otimes E$, and similarly for R_t acting on $\mathcal{S}^- \otimes E$. They both have finite propagations, and by Theorem 2.1, [29], they are both smooth in all variables. It follows immediately that S_t^2, R_t^2, $R_t D^+$, S_t and R_t are also smooth in all variables. Since propagation is additive for compositions, they all have finite propagations, which are bounded by multiples of $\sqrt{\beta(t)/\alpha(t)}$. Finally, since $\chi^t(z)$ is a Schwartz function, $\chi^t(D)$ and $\chi^t(D)D$ are bounded leafwise smoothing operators.

To deal with Q_t^-, we show that

$$\widetilde{Q}_t(D^-D^+) \;=\; \frac{\big[\big(1-\chi^t(0)^{-1}\chi^t(z)\big)\,(D)\big]^+}{D^-D^+}$$

has finite propagation which is bounded by a multiple of $\sqrt{\beta(t)/\alpha(t)}$, so also does $Q_t^- = \widetilde{Q}_t(D^-D^+)D^-$, and that $S_t Q_t^-$ is smooth in all variables and is a bounded leafwise smoothing operator.

For $u \in (0,1]$, set

$$\chi^{t,u}(z) \;=\; \left[\frac{1}{\sqrt{\alpha(t)}}\widehat{\psi}_{\frac{1}{\sqrt{\alpha(t)}}} \star e_u\right]_{\sqrt{\beta(t)}}(z),$$

and

$$\widetilde{q}^{t,u}(z) \;=\; \frac{1-\chi^{t,u}(0)^{-1}\chi^{t,u}(z)}{z^2} \;=\; \chi^{t,u}(0)^{-1}\frac{\chi^{t,u}(0)-\chi^{t,u}(z)}{z^2}.$$

Notice that $\chi^{t,0}(z)$ and $\widetilde{q}^{t,0}(z)$ are also well defined, and that for fixed z, the resulting function is continuous on $[0,1]$ and smooth on $(0,1)$. Since

$$\chi^{t,u}(z) \;=\; FT^{-1}(FT\left(\frac{1}{\sqrt{\alpha(t)}}\widehat{\psi}_{\frac{1}{\sqrt{\alpha(t)}}} \star e_u\right))(\sqrt{\beta(t)}z),$$

we have,

$$\begin{aligned}\chi^{t,u}(z) &= \int_{\mathbb{R}} \psi(\sqrt{\alpha(t)}y)\frac{1}{u}e^{-y^2/2u^2}\cos(y\sqrt{\beta(t)}z)dy\\ &= \int_{\mathbb{R}} \psi(\sqrt{\alpha(t)}uy)e^{-y^2/2}\cos(uy\sqrt{\beta(t)}z)dy,\end{aligned}$$

and,

$$\chi^{t,u}(0) \;=\; \int_{\mathbb{R}} \psi(\sqrt{\alpha(t)}uy)e^{-y^2/2}dy.$$

The latter is smooth in t and u, positive, and bounded by $\int_{\mathbb{R}} e^{-y^2/2}dy$. Thus $\chi^{t,u}(0)^{-1}$ is smooth on $(0,1)\times(0,1)$. In addition,

$$-\frac{\partial}{\partial u}\left(\chi^{t,u}(0)^{-1}\right) \;=\; \chi^{t,u}(0)^{-2}\int_{\mathbb{R}} \psi'(\sqrt{\alpha(t)}uy)\sqrt{\alpha(t)}ye^{-y^2/2}dy,$$

so has the same properties as $\chi^{t,u}(0)^{-1}$.

Next, we have

$$\widetilde{q}^{t,u}(z) \;=\; \chi^{t,u}(0)^{-1}\int_{\mathbb{R}} \psi(\sqrt{\alpha(t)}uy)e^{-y^2/2}\frac{(1-\cos(uy\sqrt{\beta(t)}z))}{z^2}dy.$$

So,

$$\begin{aligned}\frac{\partial\widetilde{q}^{t,u}}{\partial u}(z) \;=\; &\frac{\partial}{\partial u}\left(\chi^{t,u}(0)^{-1}\right)\int_{\mathbb{R}} \psi(\sqrt{\alpha(t)}uy)e^{-y^2/2}\frac{(1-\cos(uy\sqrt{\beta(t)}z))}{z^2}dy +\\ &\chi^{t,u}(0)^{-1}\int_{\mathbb{R}} \sqrt{\alpha(t)}\psi'(\sqrt{\alpha(t)}uy)ye^{-y^2/2}\frac{(1-\cos(uy\sqrt{\beta(t)}z))}{z^2}dy +\end{aligned}$$

$$\chi^{t,u}(0)^{-1} \int_{\mathbb{R}} \sqrt{\beta(t)}\psi(\sqrt{\alpha(t)}uy)ye^{-y^2/2}\frac{sin(uy\sqrt{\beta(t)}z)}{z}dy.$$

For t and z fixed, this is a smooth function of u.

Note that $\chi^{t,0}(z)$ is well defined and equals $\int_{\mathbb{R}} e^{-y^2/2}dy$, which is independent of z. Thus, $\widetilde{q}^{t,0}(z)$ is also well defined and equals 0. As $\widetilde{q}^{t,1}(D)^+ = \widetilde{Q}_t(D^-D^+)$, we have

$$\widetilde{Q}_t(D^-D^+) = \widetilde{q}^{t,1}(D)^+ - \widetilde{q}^{t,0}(D)^+ = \left[\int_0^1 \frac{\partial \widetilde{q}^{t,u}}{\partial u}(D)\,du\right]^+,$$

so we need to show that $\dfrac{\partial \widetilde{q}^{t,u}}{\partial u}(D)$ has finite propagation which is bounded by a multiple of $\sqrt{\beta(t)/\alpha(t)}$. Since $\chi^{t,u}(0)$ and $\frac{\partial}{\partial u}\left(\chi^{t,u}(0)^{-1}\right)$ are independent of z, they give multiples of the identity map when evaluated at D, so have propagation zero and may be disregarded. Thus, we may assume that $\widetilde{q}^{t,u}(z) = \dfrac{\chi^{t,u}(z) - \chi^{t,u}(0)}{z^2}$.

Since $\chi^{t,u}(z)$ is an even function, it has Taylor expansion in z with integral remainder

$$\chi^{t,u}(z) = \chi^{t,u}(0) + \frac{(\chi^{t,u})^{(2)}(0)}{2}z^2 + \frac{z^4}{6}\int_0^1 (1-v)^3(\chi^{t,u})^{(4)}(vz)dv.$$

So the Taylor expansion in z with integral remainder of $\widetilde{q}^{t,u}(z) = \dfrac{\chi^{t,u}(z) - \chi^{t,u}(0)}{z^2}$ is

$$\widetilde{q}^{t,u}(z) = \frac{(\chi^{t,u})^{(2)}(0)}{2} + \frac{z^2}{6}\int_0^1 (1-v)^3(\chi^{t,u})^{(4)}(vz)dv.$$

The term $\frac{\partial}{\partial u}((\chi^{t,u})^{(2)}(0))$ is independent of z, so, as above, it may be disregarded. Using the fact that

$$\chi^{t,u}(vz) = \int_{\mathbb{R}} \psi(\sqrt{\alpha(t)}y)\frac{1}{u}e^{-y^2/2u^2}\cos(y\sqrt{\beta(t)}vz)dy,$$

we have

$$(\chi^{t,u})^{(4)}(vz) = \int_{\mathbb{R}} \psi(\sqrt{\alpha(t)}y)\beta(t)^2v^4y^4\frac{1}{u}e^{-y^2/2u^2}\cos(y\sqrt{\beta(t)}vz)dy =$$

$$\left[\frac{1}{\sqrt{\alpha(t)}}\widehat{\psi}_{\frac{1}{\sqrt{\alpha(t)}}} \star \beta(t)^2v^4e_u^{(4)}\right](\sqrt{\beta(t)}vz) = \left[\frac{1}{\sqrt{\alpha(t)}}\widehat{\psi}_{\frac{1}{\sqrt{\alpha(t)}}} \star \beta(t)^2v^4pe_u\right](\sqrt{\beta(t)}vz),$$

where p is a finite polynomial in u and z, since $e_u(z) = e^{-u^2z^2/2}$. Note carefully that $\frac{\partial}{\partial u}(\chi^{t,u})^{(4)}(vz)$ has the same form.

As pe_u is a Schwartz function for non-zero u, so is $(\chi^{t,u})^{(4)}(vD)$, and the now usual argument shows that it has propagation $\leq v\sqrt{\beta(t)/\alpha(t)}$. As D^2 has zero propagation, $\dfrac{\partial \widetilde{q}^{t,u}}{\partial u}(D)$ has propagation which is a multiple of $\sqrt{\beta(t)/\alpha(t)}$, so also do $\widetilde{Q}_t$ and Q_t^-.

For smoothness and bounded leafwise smoothing of $S_tQ_t^-$, first note that

$$S_t \int_0^1 \frac{\partial}{\partial u}\left(\frac{(\chi^{t,u})^{(2)}(0)}{2}\right)(D)du \;=\; \left(\int_0^1 \frac{\partial}{\partial u}\left(\frac{(\chi^{t,u})^{(2)}(0)}{2}\right)(D)du\right) S_t,$$

and S_t has these properties. Finally, any positive power of D times an operator of the form

$$\left[\frac{1}{\sqrt{\alpha(t)}}\widehat{\psi}_{\frac{1}{\sqrt{\alpha(t)}}} \star \beta(t)^2 v^4 pe_u\right](\sqrt{\beta(t)}vD)$$

has these properties, (the function in the brackets is Schwartz), see for instance Theorem 2.1, [29]. Thus,

$$\int_0^1 \frac{D^2}{6}\int_0^1 (1-v)^3 \frac{\partial}{\partial u}(\chi^{t,u})^{(4)}(vD)dv\,du$$

has these properties. Therefore, $S_tQ_t^-$ has all the requisite properties, so A_t does also.

The operator δA_t is essentially a polynomial in A_t, $\partial_\nu(A_t) = [\nabla^\nu, A_t]$, and $\theta = (\nabla^\nu)^2$. Both ∇^ν and θ are smooth and bounded in all variables and are differential operators. Since A_t has finite propagation and is smooth in all variables, δA_t and $(\delta A_t)^2$ also have finite propagations and are smooth in all variables.

It remains to show that δA_t is bounded leafwise smoothing, but this is a routine exercise. We give some details for the convenience of the reader. Every term of δA_t contains either A_t, $\partial_\nu(A_t)$, or both. As A_t is bounded leafwise smoothing, we need only show that $\partial_\nu(A_t) = [\nabla^\nu, A_t]$ is bounded leafwise smoothing, since θ composed with a bounded leafwise smoothing operator is bounded leafwise smoothing. As ∂_ν is a derivation, we need only show that ∂_ν applied to the individual elements of A_t, save D^+, yields a bounded leafwise smoothing operator.

First,

$$\partial_\nu(\chi^t(D)) \;=\; \partial_\nu\left(\int_{\mathbb{R}} \frac{1}{\sqrt{\alpha(t)}}\widehat{\psi}_{\frac{1}{\sqrt{\alpha(t)}}}(y)e^{-(\sqrt{\beta(t)}D-y)^2/2}dy\right) \;=$$

$$-\frac{1}{2}\int_{\mathbb{R}} \frac{1}{\sqrt{\alpha(t)}}\widehat{\psi}_{\frac{1}{\sqrt{\alpha(t)}}}(y)\int_0^1 e^{-(1-w)(\sqrt{\beta(t)}D-y)^2/2}\partial_\nu((\sqrt{\beta(t)}D-y))^2)e^{-w(\sqrt{\beta(t)}D-y)^2/2}dw\,dy.$$

For the second equality, we refer to the proof of Proposition 3.5 of [23], which is an extension of Proposition 2.8 of [10] to foliations with Hausdorff holonomy groupoids. Now, $\partial_\nu((\sqrt{\beta(t)}D - y)^2)$ is a differential operator with smooth bounded coefficients, so $\partial_\nu(\chi^t(D))$ has the same properties as $\chi^t(D)$, i.e. it is bounded and leafwise smoothing. Thus $\partial_\nu(S_t)$ and $\partial_\nu(R_t)$ are bounded leafwise smoothing. Since $\partial_\nu(D)$ is a differential operator with smooth bounded coefficients, $R_t\partial_\nu(D)$ is also bounded leafwise smoothing. Finally, as $Q_t^- = \widetilde{Q}_t(D^-D^+)D^-$, it suffices to show that $S_t\partial_\nu(\widetilde{Q}_t(D^-D^+))$ is bounded leafwise smoothing. As above, this follows if we show that $S_t\partial_\nu(\frac{\partial q^{t,u}}{\partial u}(D))$ is bounded leafwise smoothing. For the terms

$$S_t\partial_\nu\left(\frac{\partial}{\partial u}((\chi^{t,u})^{(2)}(0))(D)\right) \;=\; \partial_\nu\left(\frac{\partial}{\partial u}((\chi^{t,u})^{(2)}(0))\right)S_t, \quad \text{and} \quad S_t\partial_\nu(\frac{D^2}{6})$$

this is obvious. As noted above, the term $\partial_\nu\left(\int_0^1 (1-v)^3\frac{\partial}{\partial u}(\chi^{t,u})^{(4)}(vD)dv\right)$ has the form

$$\partial_\nu\left(\int_0^1 (1-v)^3\left[\frac{1}{\sqrt{\alpha(t)}}\widehat{\psi}_{\frac{1}{\sqrt{\alpha(t)}}} \star \beta(t)^2v^4pe_u\right](\sqrt{\beta(t)}vD)dv\right) \;=$$

$$\partial_\nu\left(\int_0^1 (1-v)^3\int_R \Big(\frac{1}{\sqrt{\alpha(t)}}\widehat{\psi}_{\frac{1}{\sqrt{\alpha(t)}}}(y)\Big)\beta(t)^2v^4p(u,\sqrt{\beta(t)}vD-y)e^{-u(\sqrt{\beta(t)}vD-y)^2/2}dy\,dv\right).$$

The argument used for $\partial_\nu(\chi^t(D))$ is also valid here, so we have the result. □

We have the same results for D', and since A_t and A'_t are constructed directly from D and D' and have finite propagation, they are Φ compatible, as are δA_t and $\delta A'_t$. Thus $\mathrm{Tr}\left(A_t\exp(-(\delta A_t)^2\right)$ and $\mathrm{Tr}\left(A'_t\exp(-(\delta A'_t)^2\right)$ are φ compatible. Now Theorem 4.2 will be deduced right away from the following

Theorem 5.5. *For $t\in(0,1)$, the φ compatible Haefliger forms $\mathrm{Tr}\left(A_t\exp(-(\delta A_t)^2\right)$ and $\mathrm{Tr}\left(A'_t\exp(-(\delta A'_t)^2\right)$ are closed. In addition,*

$$\left[\mathrm{Tr}\left(A_t\exp(-(\delta A_t)^2)\right),\mathrm{Tr}\left(A'_t\exp(-(\delta A'_t)^2)\right)\right] \;\in\; H_c^*(M/F,M'/F';\varphi)$$

is independent of t. So $\mathrm{ch}(\mathrm{Ind}_a(D,D'))$ is well defined, and

$$\mathrm{ch}(\mathrm{Ind}_a(D,D')) \;=\; \mathrm{Ind}_t(D,D').$$

Proof. The Haefliger forms are closed by Proposition 3.1, which also gives that $\frac{d}{dt}\mathrm{Tr}\left(A_t\exp(-(\delta A_t)^2\right) = d_HW_t$, and $\frac{d}{dt}\mathrm{Tr}\left(A'_t\exp(-(\delta A'_t)^2\right) = d_HW'_t$. To finish the

proof of t independence, we need only show that W_t and W'_t can be chosen to be φ compatible.

Recall that $\pi_\pm$ is the grading operator $\pi_\pm = \mathrm{diag}(\mathrm{I}_{\mathcal{S}^+\otimes E}, -\mathrm{I}_{\mathcal{S}^-\otimes E}) = \pi_+ - \pi_-$, and similarly for $\pi'_\pm$. When we identify the spin bundles and Dirac operators off compact subspaces, we also identify these gradings, so they are Φ compatible. In particular, π_- and π'_- are Φ compatible. Note that $A_t + \pi_-$ and $A'_t + \pi'_-$ are idempotents. Using this fact, in [4], Corrigendum, it is shown that

$$\frac{d}{dt}\left(\mathrm{Tr}\left((A_t+\pi_-)\exp(-(\delta(A_t+\pi_-))^2)\right), \mathrm{Tr}\left((A'_t+\pi'_-)\exp(-(\delta(A'_t+\pi'_-)^2))\right)\right) = d_H(W_t, W'_t),$$

where $(W_t, W'_t) \in \mathcal{A}^*_c(M/F, M'/F'; \varphi)$, in particular they are φ compatible. This follows from the fact that the operators $\partial_\nu, \theta, \Theta$, and δ all preserve Φ compatibility, and that W_t and W'_t are constructed using those operators, A_t, A'_t, π_-, π'_- (and the identities I and I$'$), and their derivatives with respect to t. Since A_t, A'_t, π_-, π'_-, I and I$'$ are Φ compatible, W_t and W'_t are φ compatible. As

$$\begin{aligned} d_H \,\mathrm{Tr}\left((A_t+\pi_-)\exp(-(\delta(A_t+\pi_-))^2)\right) &= d_H \,\mathrm{Tr}\left((A'_t+\pi'_-)\exp(-(\delta(A'_t+\pi'_-)^2)\right) \\ &= 0, \end{aligned}$$

it follows that

$$\left[\mathrm{Tr}\left((A_t+\pi_-)\exp(-(\delta(A_t+\pi_-))^2)\right), \mathrm{Tr}\left((A'_t+\pi'_-)\exp(-(\delta(A'_t+\pi'_-)^2)\right)\right] \in H^*_c(M/F, M'/F'; \varphi)$$

is independent of t.

Next, using Proposition 3.5 and Corollary 3.7 of [5], with the reasoning above, (that is: all the operators used in the proofs preserve Φ compatibility, so if the input is Φ compatible, the output is φ compatible), shows that

$$\begin{aligned} &\left[\mathrm{Tr}\left(A_t\exp(-(\delta A_t)^2)\right), \mathrm{Tr}\left(A'_t\exp(-(\delta A'_t)^2)\right)\right] = \\ &\left[\mathrm{Tr}\left((A_t+\pi_-)\exp(-(\delta(A_t+\pi_-))^2)\right), \mathrm{Tr}\left((A'_t+\pi'_-)\exp(-(\delta(A'_t+\pi'_-)^2)\right)\right] \\ &\in H^*_c(M/F, M'/F'; \varphi). \end{aligned}$$

For the equality $\mathrm{ch}(\mathrm{Ind}_a(D, D')) = \mathrm{Ind}_t(D, D')$, standard techniques used in [24,5], coupled with Remark 5.2, show that

$$\lim_{t\to 0} \mathrm{tr}\left(A_t\exp(-(\delta A_t)^2)\right) = AS(D_F),$$

uniformly pointwise on M, and we have the same for A'_t. As $\mathrm{Tr}\left(A_t\exp(-(\delta A_t)^2)\right)$ involves integrating over compact subsets, we may interchange the limit with the integration. □

So we have Theorem 4.2.

Note that so far, we have not used the assumptions in Theorem 4.3 or 4.6. We now move on to the proofs of Theorems 4.3 and 4.5. For the proof of Theorem 4.3, we need to show that

$$\lim_{t\to 1} \mathrm{Tr}\left(A_t \exp(-(\delta A_t)^2)\right) \;=\; \mathrm{Tr}\left(\pi_\pm P_0 \exp(-(\delta(\pi_\pm P_0))^2)\right).$$

Recall that A_t is only defined for $0 < t < 1$, that P_0 is the projection onto the kernel of D^2, and that $A_t \exp(-(\delta A_t)^2)$ has propagation bounded by $c_A\sqrt{\beta(t)/\alpha(t)}$ for some $c_A \in \mathbb{R}^+$. We recall below that

$$\lim_{t\to 1} k_{P_{[0,\epsilon)}A_tP_{[0,\epsilon)})}(\overline{x},\overline{x}) \;=\; k_{P_{[0,\epsilon)}\pi_\pm P_0 P_{[0,\epsilon)}}(\overline{x},\overline{x}) \;=\; k_{\pi_\pm P_0}(\overline{x},\overline{x}),$$

uniformly pointwise which is sufficient for our purposes.

Denote by $\varrho_{[\epsilon,\infty)}$ the characteristic function for the interval $[\epsilon,\infty)$.

Lemma 5.6. *For ℓ a non-negative integer, there exists a constant $C_\ell > 0$ depending only on ℓ, such that*

$$||z^\ell \varrho_{[\epsilon,\infty)}\chi^t(z)\varrho_{[\epsilon,\infty)}||_\infty \le C_\ell e^{-1/64\alpha(t)} + \epsilon^\ell e^{-\epsilon^2\beta(t)/2} \;\to\; 0, \;\; \textit{exponentially as } t \to 1.$$

Proof. First note that,

$$||z^\ell \varrho_{[\epsilon,\infty)}\chi^t\varrho_{[\epsilon,\infty)}||_\infty \;\le\; ||z^\ell\varrho_{[\epsilon,\infty)}\left(\chi^t - e_{\sqrt{\beta(t)}}\right)\varrho_{[\epsilon,\infty)}||_\infty + ||z^\ell\varrho_{[\epsilon,\infty)}e_{\sqrt{\beta(t)}}\varrho_{[\epsilon,\infty)}||_\infty \;\le$$
$$||z^\ell\varrho_{[\epsilon,\infty)}\left(\chi^t - e_{\sqrt{\beta(t)}}\right)\varrho_{[\epsilon,\infty)}||_\infty + \epsilon^\ell e^{-\epsilon^2\beta(t)/2},$$

since the maximum for the second term for t close enough to 1 will occur at $z = \epsilon$, as $\beta(t) \to \infty$ as $t \to 1$.

Next, $||z^\ell\varrho_{[\epsilon,\infty)}\left(\chi^t - e_{\sqrt{\beta(t)}}\right)\varrho_{[\epsilon,\infty)}||_\infty$ is bounded by $||z^\ell\left(\chi^t - e_{\sqrt{\beta(t)}}\right)||_\infty$, which in turn is bounded by the L^1 norm of $FT(z^\ell\left(\chi^t - e_{\sqrt{\beta(t)}}\right))$. Up to a constant depending only on ℓ,

$$FT(z^\ell\left(\chi^t - e_{\sqrt{\beta(t)}}\right)) \;=\; \frac{\partial^\ell}{\partial z^\ell}FT\left(\chi^t - e_{\sqrt{\beta(t)}}\right) =$$
$$\frac{\partial^\ell}{\partial z^\ell}\left(\frac{1}{\sqrt{\beta(t)}}FT\left(\frac{1}{\sqrt{\alpha(t)}}\hat{\psi}_{1/\sqrt{\alpha(t)}} * e\right)_{1/\sqrt{\beta(t)}} - \frac{1}{\sqrt{\beta(t)}}e_{1/\sqrt{\beta(t)}}\right) =$$
$$\frac{\partial^\ell}{\partial z^\ell}\left(\frac{1}{\sqrt{\beta(t)}}\left(\psi_{\sqrt{\alpha(t)}}e\right)_{1/\sqrt{\beta(t)}} - \frac{1}{\sqrt{\beta(t)}}e_{1/\sqrt{\beta(t)}}\right) \;=\; \frac{\partial^\ell}{\partial z^\ell}\left(\frac{e_{1/\sqrt{\beta(t)}}}{\sqrt{\beta(t)}}\left(\psi_{\sqrt{\frac{\alpha(t)}{\beta(t)}}} - 1\right)\right) =$$
$$\sum_{k=0}^{\ell}\binom{\ell}{k}\frac{\partial^k}{\partial z^k}\left(\frac{e_{1/\sqrt{\beta(t)}}}{\sqrt{\beta(t)}}\right)\frac{\partial^{\ell-k}}{\partial z^{\ell-k}}\left(\psi_{\sqrt{\frac{\alpha(t)}{\beta(t)}}} - 1\right).$$

The function $\psi_{\sqrt{\frac{\alpha(t)}{\beta(t)}}} - 1 = 0$ on $|z| \leq \frac{1}{4\sqrt{\alpha(t)/\beta(t)}}$, and the norms of its derivatives are globally bounded by a constant depending only on ℓ. Thus the L^1 norm of $FT\left(z^\ell\left(\chi^t - e_{\sqrt{\beta(t)}}\right)\right)$ is bounded by a constant, depending only on ℓ, times

$$\sum_{k=0}^{\ell} \int_{|z|\geq \frac{1}{4\sqrt{\alpha(t)/\beta(t)}}} \left|\frac{\partial^k}{\partial z^k}\left(e_{1/\sqrt{\beta(t)}}\right)\right| \frac{dz}{\sqrt{\beta(t)}}$$

$$= \sum_{k=0}^{\ell} \int_{|z|\geq \frac{1}{4\sqrt{\alpha(t)/\beta(t)}}} \left|p_k(1/\sqrt{\beta}(t), z/\sqrt{\beta})\right| \left(e_{1/\sqrt{\beta(t)}}\right) \frac{dz}{\sqrt{\beta(t)}} =$$

$$\sum_{k=0}^{\ell} \int_{|z|\geq \frac{1}{4\sqrt{\alpha(t)}}} \left|p_k(1/\sqrt{\beta}(t), z)\right| e^{-z^2/2} dz \;\leq\; e^{-1/64\alpha(t)} \int_{\mathbb{R}} \sum_{k=0}^{\ell} \left|p_k(1/\sqrt{\beta}(t), z)\right| e^{-z^2/4} dz.$$

Here p_k is a polynomial of degree k in both variables, so the integral is bounded by a constant depending only on ℓ. Since $\alpha(t) \to 0$ and $\beta(t) \to \infty$ as $t \to 1$, we have the lemma. □

Denote by Q_ϵ the spectral projection for D^2 for the interval $[\epsilon, \infty)$, that is $Q_\epsilon = \varrho_{[\epsilon,\infty)}(D^2)$. Since $I = P_{[0,\epsilon)} + Q_\epsilon$, the operators Q_ϵ and δQ_ϵ are transversely smooth and bounded, as the other two operators are because of our assumption of transverse smoothness. The operators $P_{[0,\epsilon)}$, Q_ϵ, and A_t all commute as they are functions of D, so

$$A_t \exp\left(-(\delta A_t)^2\right) = P_{[0,\epsilon)} A_t P_{[0,\epsilon)} \exp\left(-(\delta A_t)^2\right) \;+\; Q_\epsilon A_t Q_\epsilon \exp\left(-(\delta A_t)^2\right).$$

Recall that δA_t is nilpotent, in particular, $\exp\left(-(\delta A_t)^2\right) = \sum_{k=0}^{q/2} \frac{(-(\delta A_t)^2)^k}{k!}$, see [5]. Lemma 5.6 gives immediately that $||D^{2\ell} Q_\epsilon \chi^t(D) Q_\epsilon|| \to 0$ exponentially as $t \to 1$. The fact that every element of A_t contains at least one $\chi^t(D)$, and that all the other terms are bounded, save D^+ (but $R_t D^+$ is covered by Lemma 5.6), give that $||D^{2\ell} Q_\epsilon A_t Q_\epsilon|| \to 0$ exponentially as $t \to 1$. Thus, $||D^{2\ell} Q_\epsilon A_t Q_\epsilon \exp\left(-(\delta A_t)^2\right)|| \to 0$ exponentially as $t \to 1$. It follows from the proof of Theorem 2.3.9 and the statement of Theorem 2.3.13, both of [24], that the Schwartz kernel of $Q_\epsilon A_t Q_\epsilon \exp\left(-(\delta A_t)^2\right) \to 0$ pointwise uniformly exponentially as $t \to 1$. So,

$$\lim_{t\to 1} \mathrm{Tr}\left(Q_\epsilon A_t Q_\epsilon \exp\left(-(\delta A_t)^2\right)\right) \;=\; 0,$$

in $\mathcal{A}_c^*(M/F)$ and similarly for $Q'_\epsilon A'_t Q'_\epsilon$. Thus we may ignore those terms. Note carefully that this is true for *fixed* $\epsilon > 0$.

For the terms coming from $P_{[0,\epsilon)} A_t P_{[0,\epsilon)}$, note that for t near 1, $2\varrho_{[0,\epsilon)}(z)$ dominates $\varrho_{[0,\epsilon)} \chi^t \varrho_{[0,\epsilon)}(z)$. This follows from Remark 5.2, since $\lim_{t\to 1} \chi^t(z) = e^{-z^2/(2(1-t))}$ in the

Schwartz topology, and for t near 1, $\sup_z e^{-z^2/(2(1-t))} = 1$. Thus, for ℓ a fixed positive integer and for t near 1,

$$2\epsilon^\ell ||\varrho_{[0,\epsilon)}(z)||_\infty \;=\; 2||z^\ell \varrho_{[0,\epsilon)}(z)||_\infty \;\geq\; ||z^\ell \varrho_{[0,\epsilon)}\chi^t \varrho_{[0,\epsilon)}(z)||_\infty.$$

The fact that $||\delta A_t||$ and $||\exp\left(-(\delta A_t)^2\right)||$ are bounded and the argument above imply that a multiple of $\mathrm{tr}\left(k_{P_{[0,\epsilon)}}(\overline{x},\overline{x})\right)$ dominates $||\,\mathrm{tr}\Big(k_{P_{[0,\epsilon)}A_tP_{[0,\epsilon)}\exp(-(\delta A_t)^2)}(\overline{x},\overline{x})\Big)||$. Since we can ignore $Q_\epsilon A_t Q_\epsilon \exp\left(-(\delta A_t)^2\right)$, the Dominated Convergence Theorem implies

$$\lim_{t\to 1}\mathrm{Tr}\Big(A_t\exp\left(-(\delta A_t)^2\right)\Big) \;=\; \lim_{t\to 1}\mathrm{Tr}\Big(P_{[0,\epsilon)}A_tP_{[0,\epsilon)}\exp\left(-(\delta A_t)^2\right)\Big) \;=$$
$$\lim_{t\to 1}\int\limits_F \mathrm{tr}\Big(P_{[0,\epsilon)}A_tP_{[0,\epsilon)}\exp\left(-(\delta A_t)^2\right)\Big) \;=\; \int\limits_F \lim_{t\to 1}\mathrm{tr}\Big(P_{[0,\epsilon)}A_tP_{[0,\epsilon)}\exp\left(-(\delta A_t)^2\right)\Big),$$

and similarly for A'_t.

The proof of Theorem 4.2 in [5], which requires that F be Riemannian, shows that, under our conditions on the Novikov-Shubin invariants, in degree $2k$ for $0 \leq 2k \leq 2\ell$ we have,

$$\lim_{t\to 1}(P_{[0,\epsilon)}A_tP_{[0,\epsilon)}) \;=\; P_{[0,\epsilon)}\pi_\pm P_0 P_{[0,\epsilon)} \;=\; \pi_\pm P_0,$$

uniformly pointwise, and similarly for A'_t. So, in degree $2k$ for $0 \leq 2k \leq 2\ell$,

$$\int\limits_F \lim_{t\to 1}\mathrm{tr}\Big(P_{[0,\epsilon)}A_tP_{[0,\epsilon)}\exp\left(-(\delta A_t)^2\right)\Big) \;=\; \int\limits_F \mathrm{tr}\Big(\pi_\pm P_0\exp\left(-(\delta(\pi_\pm P_0))^2\right)\Big) \;=\; \mathrm{ch}_a(P_0).$$

So, we have proven Theorems 4.3 and 4.5.

It remains to prove Theorem 4.6, and we thus need to compute the limits as $t \to 0$ and $t \to 1$ of

$$\lim_{s\to\infty}\left(\int\limits_{T\smallsetminus T_s}\mathrm{Tr}\left(A_t\exp(-(\delta A_t)^2\right)\wedge\omega_T \;-\int\limits_{T'\smallsetminus T'_s}\mathrm{Tr}\left(A'_t\exp(-(\delta A'_t)^2\right)\wedge\omega'_{T'}\right).$$

For $\lim_{t\to 0}$, we may assume that the two integrands agree on $M(0) = V_M$ and $M'(0) = V'_{M'}$ (actually on fixed penumbras). Then we have,

$$\lim_{t\to 0}\lim_{s\to\infty}\left(\int\limits_{T\smallsetminus T_s}\mathrm{Tr}\left(A_t\exp(-(\delta A_t)^2\right)\wedge\omega_T \;-\int\limits_{T'\smallsetminus T'_s}\mathrm{Tr}\left(A'_t\exp(-(\delta A'_t)^2\right)\right)\wedge\omega'_{T'}\right) \;=$$
$$\lim_{t\to 0}\left(\int\limits_{M\smallsetminus M(0)}\mathrm{tr}\left(A_t\exp(-(\delta A_t)^2\right)\wedge\omega \;-\int\limits_{M'\smallsetminus M'(0)}\mathrm{tr}\left(A'_t\exp(-(\delta A'_t)^2\right)\right)\wedge\omega'\right) \;=$$

$$\int_{\mathcal{K}} AS(D_F)\wedge\omega - \int_{\mathcal{K}'} AS(D'_{F'})\wedge\omega' \;=\; \left\langle \left(\left[\int_F \mathrm{AS}(D_F), \int_{F'} AS(D_{F'}) \right] \right), (\omega_T, \omega'_{T'}) \right\rangle .$$

As above, $\lim_{t\to 0} \mathrm{tr}\left(A_t \exp(-(\delta A_t)^2)\right) \;=\; AS(D_F)$, uniformly pointwise on M, and we have the same for A'_t. As we are integrating over compact subsets, we may interchange the $\lim_{t\to 0}$ with the integrations.

For $\lim_{t\to 1}$, note that the operators have propagations bounded by $c_A\sqrt{\beta(t)/\alpha(t)}$ for some $c_A \in \mathbb{R}^+$. As they are Φ compatible, we may assume that the two integrands agree on $T_{c_A(1-t)^{-1}}$ and $T'_{c_A(1-t)^{-1}}$. Thus,

$$\lim_{t\to 1}\lim_{s\to\infty} \left(\int_{T\smallsetminus T_s} \mathrm{Tr}\left(A_t \exp(-(\delta A_t)^2\right)\wedge\omega_T \;-\; \int_{T'\smallsetminus T'_s} \mathrm{Tr}\left(A'_t \exp(-(\delta A'_t)^2)\right)\wedge\omega'_{T'} \right) \;=$$

$$\lim_{t\to 1} \left(\int_{T\smallsetminus T_{c_A(1-t)^{-1}}} \mathrm{Tr}\left(A_t \exp(-(\delta A_t)^2\right)\wedge\omega_T \;-\; \int_{T'\smallsetminus T'_{c_A(1-t)^{-1}}} \mathrm{Tr}\left(A'_t \exp(-(\delta A'_t)^2)\right)\wedge\omega'_{T'} \right).$$

Since the Schwartz kernel of $Q_\epsilon A_t Q_\epsilon \exp\left(-(\delta A_t)^2\right) \to 0$ pointwise exponentially as $t \to 1$, the fact that ω_T is bounded, and the assumption that M has sub-exponential growth, give that

$$\lim_{t\to 1} \left(\int_{T\smallsetminus T_{c_A(1-t)^{-1}}} \mathrm{Tr}\Big(Q_\epsilon A_t Q_\epsilon \exp\left(-(\delta A_t)^2\right)\Big)\wedge\omega_T \right) \;=\; 0,$$

and similarly for $Q'_\epsilon A'_t Q'_\epsilon$. Thus we may ignore those terms.

Next, we have that for t near 1, a multiple of $\int_M \mathrm{tr}\left(k_{P_{[0,\epsilon)}}\right)$ dominates

$$|| \int_M \mathrm{tr}\Big(k_{P_{[0,\epsilon)} A_t P_{[0,\epsilon)} \exp(-(\delta A_t)^2)}\Big) ||.$$

But this latter equals $|| \int_M \mathrm{tr}\Big(k_{P_{[0,\epsilon)} A_t P_{[0,\epsilon)} \exp(-(\delta A_t)^2) P_{[0,\epsilon)}}\Big) ||$, since $\int_M \mathrm{tr} \;=\; \int_T \int_F \mathrm{tr} \;=\; \int_T \mathrm{Tr}$, and Tr is a trace. Thus, we need only show that a multiple of

$$\mathrm{tr}\left(k_{P_{[0,\epsilon)}}(\overline{x},\overline{x})\right) \text{ dominates } || \,\mathrm{tr}\left(k_{P_{[0,\epsilon)} A_t P_{[0,\epsilon)} \exp(-(\delta A_t)^2) P_{[0,\epsilon)}}(\overline{x},\overline{x})\right) ||.$$

This is due to the fact that, for a smoothing operator A, $\mathrm{tr}(k_A((\overline{x},\overline{x})) = \sum_i \langle A(\delta^{\overline{x}}_{v_i}), \delta^{\overline{x}}_{v_i}\rangle$. Here v_i is an orthonormal basis of the fiber over the point $\overline{x}$, and $\delta^{\overline{x}}_{v_i}$ is the Dirac delta section of the bundle supported at $\overline{x}$. Furthermore, everything is well defined on bounded geometry manifolds. See, for example, [24] for details of such arguments. As the operators we are concerned with are bounded leafwise smoothing, we have,

$$\begin{aligned} &||\langle P_{[0,\epsilon)} A_t P_{[0,\epsilon)} \exp\left(-(\delta A_t)^2\right) P_{[0,\epsilon)}(\delta^{\overline{x}}_{v_i}), \delta^{\overline{x}}_{v_i}\rangle|| \\ &= ||\langle A_t P_{[0,\epsilon)} \exp\left(-(\delta A_t)^2\right) P_{[0,\epsilon)}(\delta^{\overline{x}}_{v_i}), P_{[0,\epsilon)}(\delta^{\overline{x}}_{v_i})\rangle|| \ \leq \\ ||A_t P_{[0,\epsilon)} \exp\left(-(\delta A_t)^2\right)||\, ||P_{[0,\epsilon)}(\delta^{\overline{x}}_{v_i})||^2 \ &= \ ||A_t P_{[0,\epsilon)} \exp\left(-(\delta A_t)^2\right)||\, \langle P_{[0,\epsilon)}(\delta^{\overline{x}}_{v_i}), \delta^{\overline{x}}_{v_i}\rangle. \end{aligned}$$

Summing over i, gives the result.

The fact that $||\omega||$ is bounded and the assumption that $\int_M \mathrm{tr}\left(P_{[0,\epsilon)}\right) dx \ < \ \infty$, imply that

$$\int_M \mathrm{tr}\left(P_{[0,\epsilon)}\right) ||\omega||\, dx \ < \ \infty.$$

Thus

$$\int_M ||\,\mathrm{tr}\Big(P_{[0,\epsilon)} A_t P_{[0,\epsilon)} \exp\left(-(\delta A_t)^2\right)(\overline{x},\overline{x})\Big) \wedge \omega||\, dx \ < \ \infty,$$

so the integral $\int_M \mathrm{tr}\Big(P_{[0,\epsilon)} A_t P_{[0,\epsilon)} \exp\left(-(\delta A_t)^2\right)\Big) \wedge \omega$ converges. Notice that

$$\int_M \mathrm{tr}\Big(P_{[0,\epsilon)} A_t P_{[0,\epsilon)} \exp\left(-(\delta A_t)^2\right)\Big) \wedge \omega \ = \ \int_T \mathrm{Tr}\Big(P_{[0,\epsilon)} A_t P_{[0,\epsilon)} \exp\left(-(\delta A_t)^2\right)\Big) \wedge \omega_T.$$

This fact, the fact that we can ignore $Q_\epsilon A_t Q_\epsilon \exp\left(-(\delta A_t)^2\right)$, and the Dominated Convergence Theorem imply,

$$\begin{aligned} &\lim_{t\to 1} \int_{T \setminus T_{c_A(1-t)^{-1}}} \mathrm{Tr}\Big(A_t \exp\left(-(\delta A_t)^2\right)\Big) \wedge \omega_T \\ &= \lim_{t\to 1} \int_{T \setminus T_{c_A(1-t)^{-1}}} \mathrm{Tr}\Big(P_{[0,\epsilon)} A_t P_{[0,\epsilon)} \exp\left(-(\delta A_t)^2\right)\Big) \wedge \omega_T \ = \\ &\quad \lim_{t\to 1} \int_T \mathrm{Tr}\Big(P_{[0,\epsilon)} A_t P_{[0,\epsilon)} \exp\left(-(\delta A_t)^2\right)\Big) \wedge \omega_T \\ &\quad = \int_T \lim_{t\to 1} \mathrm{Tr}\Big(P_{[0,\epsilon)} A_t P_{[0,\epsilon)} \exp\left(-(\delta A_t)^2\right)\Big) \wedge \omega_T, \end{aligned}$$

and similarly for A'_t. The proof of Theorem 4.2 in [5] shows that, as above, under our conditions on the Novikov-Shubin invariants in Theorem 4.6,

$$\lim_{t\to 1} \mathrm{Tr}\Big(A_t \exp\left(-(\delta A_t)^2\right)\Big) \wedge \omega_T \;=\; \mathrm{Tr}\Big(\pi_\pm P_0 \exp\left(-(\delta(\pi_\pm P_0))^2\right)\Big) \wedge \omega_T,$$

and similarly for A'_t. So,

$$\lim_{t\to 1}\left(\int\limits_{T \pitchfork T_{c_A(1-t)^{-1}}} \mathrm{Tr}\Big(A_t \exp\left(-(\delta A_t)^2\right)\Big) \wedge \omega_T - \int\limits_{T' \pitchfork T'_{c_A(1-t)^{-1}}} \mathrm{Tr}\Big(A'_t \exp\left(-(\delta A'_t)^2\right)\Big) \wedge \omega'_{T'}\right) =$$

$$\int\limits_T \mathrm{Tr}\Big(\pi_\pm P_0 \exp\left(-(\delta(\pi_\pm P_0))^2\right)\Big) \wedge \omega_T \;-\; \int\limits_{T'} \mathrm{Tr}\Big(\pi'_\pm P'_0 \exp\left(-(\delta(\pi'_\pm P'_0))^2\right)\Big) \wedge \omega'_{T'} \;=$$

$$\langle (\mathrm{ch}_a(P_0), \mathrm{ch}_a(P'_0)), (\omega_T, \omega'_{T'})\rangle.$$

That is, $\langle \mathrm{ch}\,\mathrm{Ind}_a(D, D'), [\omega_T, \omega'_{T'}]\rangle = \langle (\mathrm{ch}_a(P_0), \mathrm{ch}_a(P'_0)), (\omega_T, \omega'_{T'})\rangle$. So we have proven Theorem 4.6.

Remark 5.7. Note that if M, so also M', grows exponentially, there are constants $c_0, c_M \in \mathbb{R}^+$, so that $\mathrm{vol}(M_t) \leq c_0 e^{c_M t}$. This follows from the Bishop-Gromov inequality. Thus, if we used Lemma 5.6 as above and integrated over $M \smallsetminus M_{c_A\sqrt{\beta(t)/\alpha(t)}}$, we would get an estimate of the form,

$$(C_\ell e^{-1/64\alpha(t)} + \epsilon^\ell e^{-\epsilon^2\beta(t)/2}) c_0 e^{c_M c_A \sqrt{\beta(t)/\alpha(t)}}.$$

For the proof to work, we need this to $\to 0$ as $t \to 1$. Now $\sqrt{\beta(t)/\alpha(t)} = \beta(t) = 1/\alpha(t)$, as $t \to 1$. Thus the two terms must $\to 0$ individually. This only happens if $c_M c_A < \min(\epsilon^2/2, \frac{1}{64})$. That is, the exponential growth is not too robust.

6. Invertible near infinity operators

In this section, we assume that (M, F) is as in the first two paragraphs of Section 2.

6.1. Invertibility near infinity

Our new assumption here is that the zero-th order contribution $\mathcal{R}^E_F$ in the Bochner formula defined below is strictly positive on M near infinity. As $\mathcal{R}^E_F$ is locally defined, this implies that the same for $\mathcal{R}^{E'}_{F'}$.

For the leafwise Dirac operator $D_F = (D_L)$, the canonical operator $\mathcal{R}^E_F$ on sections of $E_M|_L$ is given by

$$\mathcal{R}^E_F(\varphi) \;=\; \frac{1}{2}\sum_{i,j=1}^{p} \rho(X_i)\rho(X_j) R^E_{X_i,X_j}(\varphi),$$

where R^E is the curvature operator of the Hermitian connection $\nabla^{F,E}$ on $E_M|_L$, $X_1, \ldots, X_p$ is a local oriented orthonormal basis of TL, and $\rho(X_i)$ is the Clifford action of X_i. Note that $\mathcal{R}^E_F$ is well defined, smooth, and that it is globally bounded because of the assumption of bounded geometry. The operators D_L and $\mathcal{R}^E_F$ are related by the general leafwise Bochner Identity, [27]

$$D_L^2 = (\nabla^{F,E})^* \nabla^{F,E} \; + \; \mathcal{R}^E_F. \tag{6.1}$$

As we work on $\mathcal{G}$ rather than M, $D = r^*(D_F)$ also satisfies Equation 6.1, which, being local, is the same, namely, $D^2 = \nabla^*\nabla + r^*(\mathcal{R}^E_F)$. Note that in general, if $\mathcal{R}^E_F$ is strictly positive near infinity, $r^*(\mathcal{R}^E_F)$ is not, due to the fact that r is not a proper map in general. However, $r^*(\mathcal{R}^E_F)$ is $\mathcal{G}$-invariant strictly positive near infinity off some $\mathcal{G}$-compact subspace, in particular when restricted to $M \subset \mathcal{G}$, since it coincides with $\mathcal{R}^E_F$ there.

We have the following result from of [8]. Note that it does not need $P_{[0,\epsilon]}$ to be transversely smooth. It does need it to be transversely measurable, which it is by Lemma 4.10 of [8].

Theorem 6.2. *(Theorem 5.2 of [8]) Assume that F admits a holonomy invariant transverse measure Λ. Suppose $\mathcal{R}^E_F$ is strictly positive near infinity. In particular, we may assume that $\kappa_0 = \sup\{\kappa \in \mathbb{R} \,|\, \mathcal{R}^E_F - \kappa \,\mathrm{I} \geq 0 \text{ on } M \setminus \mathcal{K}_M\}$ is positive. Then, for $0 \leq \epsilon < \kappa_0$,*

$$\int_M \mathrm{tr}(P_{[0,\epsilon]}(\overline{x},\overline{x}))\, dx_F d\Lambda \;\leq\; \frac{(\kappa_0 - \kappa_1)}{(\kappa_0 - \epsilon)} \int_{\mathcal{K}_M} \mathrm{tr}(P_{[0,\epsilon]}(\overline{x},\overline{x}))\, dx_F d\Lambda \;<\; \infty,$$

where $\kappa_1 = \sup\{\kappa \in \mathbb{R} \,|\, \mathcal{R}^E_F - \kappa \,\mathrm{I} \geq 0 \text{ on } M\}$.

Note that if F is Riemannian, it does admit holonomy invariant transverse measures, and we can insure that dx is of the form $dx_F d\Lambda$.

Proof. The proof of Theorem 5.2 in [8] works equally well here, mutatis mutandis. The changes in notation needed are

$$D^E_L \to D,\; k_{P_{[0,\epsilon]}}(x,x) \to k_{P_{[0,\epsilon]}}(\overline{x},\overline{x}),\; L \to \widetilde{L}_x,\; \sigma_L \to \sigma_x,\; \int_L \to \int_{\widetilde{L}_x},$$

and so on. □

Proposition 5.5 in [8] still holds here, namely the following.

Proposition 6.3. *Suppose the curvature operator $\mathcal{R}^E_F$ is strictly positive on M, that is $\kappa_1 > 0$, so $\mathcal{R}^E_F \geq \kappa_1 \,\mathrm{I}$ on M. Then for $0 \leq \epsilon < \kappa_1$, $P_{[0,\epsilon]} = 0$.*

The relationship with the index bundle is not insured in general, [6], and one needs to impose additional spectral assumptions. We have, as in [8], the following immediate corollaries of Theorems 4.3, 4.6 and 6.2 which relate the pairings there to pairings with the index bundles.

Theorem 6.4. *Suppose that $(M, F, \mathcal{K}_M)$ and $(M', F', \mathcal{K}_{M'})$ are bounded geometry foliations which are identified outside the compact subspaces $\mathcal{K}_M$ and $\mathcal{K}_{M'}$ as before and let (ω, ω') be a φ-compatible pair of closed holonomy invariant forms of degree $\ell \leq q$. Assume the following:*

- *M, and so also M', has sub-exponential growth, and F and F' are Riemannian;*
- *the leafwise operators P_0, P_0', $P_{(0,\epsilon)}$ and $P'_{(0,\epsilon)}$ (for ϵ sufficiently small) are transversely smooth;*
- *$NS(D)$ and $NS(D')$ are greater than ℓ;*
- *$\mathcal{R}_F^E$, so also $\mathcal{R}_{F'}^{E'}$, is strictly positive near infinity in M and M' respectively.*

Then

$$\langle \left[\int_F AS(D_F), \int_{F'} AS(D_{F'}) \right], [\omega_T, \omega'_{T'}] \rangle \;=\; \langle (\mathrm{ch}(P_0), \mathrm{ch}(P_0'), (\omega_T, \omega'_{T'}) \rangle .$$

Recall that $(\mathrm{ch}_a(P_0), \mathrm{ch}_a(P_0')$ is not an element of $\mathcal{A}_c^*(M/F, M'/F'; \varphi)$ in general. Since $AS(D_F)$ and $AS(D_F)$ are φ compatible, $AS(D_F) \wedge \omega$ and $AS(D'_{F'}) \wedge \omega$ are φ compatible, say off the compact subsets $\widehat{\mathcal{K}}$ and $\widehat{\mathcal{K}}'$, and then we have

$$\langle \left[\int_F AS(D_F), \int_{F'} AS(D_{F'}) \right], [\omega_T, \omega'_{T'}] \rangle \;=\; \int_{\widehat{\mathcal{K}}} AS(D_F) \wedge \omega \;-\; \int_{\widehat{\mathcal{K}}'} AS(D_{F'}) \wedge \omega' .$$

For a single foliated manifold we have the following, compare with [20].

Theorem 6.5. *Suppose that E and E' are two Clifford bundles over the foliated manifold (M, F), which are isomorphic off the compact subset $\mathcal{K}_M$, with associated Dirac operators D and D'. Let ω be a bounded closed holonomy invariant transverse form (or current) of degree $\ell \leq q$. Suppose that*

- *M has sub-exponential growth, and F is Riemannian;*
- *the leafwise operators P_0, P_0', $P_{(0,\epsilon)}$ and $P'_{(0,\epsilon)}$ (for ϵ sufficiently small) are transversely smooth;*
- *$\min(NS(D), NS(D'))$ is greater than ℓ;*
- *$\mathcal{R}_F^E$, and hence also $\mathcal{R}_F^{E'}$, is strictly positive near infinity.*

Then, since $\mathrm{ch}(E) = \mathrm{ch}(E')$ *off* $\mathcal{K}_M$*,*

$$\begin{aligned}\langle \mathrm{ch}(\mathrm{Ind}_a(D, D')), [\omega_T, \omega_T]\rangle &= \int_{\mathcal{K}_M} (\mathrm{AS}(D_F)(\mathrm{ch}(E) - \mathrm{ch}(E')) \wedge \omega \\ &= \langle (\mathrm{ch}(P_0), \mathrm{ch}(P_0')), (\omega_T, \omega_T)\rangle.\end{aligned}$$

Remark 6.6. Note that if E_1 is a leafwise almost flat bundle (actually a K-theory class) on M, then we may twist the operators D and D' by E_1 to get the operators D_{E_1} and D'_{E_1}. Uniform positivity near infinity is preserved when this is done, so we have the extension of Theorem 6.5 to D_{E_1} and D'_{E_1}. Theorem 6.4 also extends in this way if we have leafwise almost flat bundles $E_1 \to M$ and $E_1' \to M'$ which are isomorphic near infinity.

6.2. Reflective foliations

We now relate our definition of the relative index to the cut-and-paste definition considered in Section 4 of [20]. For this paper to be self-contained, we paraphrase from [8]. For simplicity, we assume that ω and ω' are φ compatible off $\mathcal{K}_M$ and $\mathcal{K}'_{M'}$.

We say that $(M, F, \mathcal{K}_M)$ as above is reflective if there exists a compact submanifold $H \subset M$ such that

$$\mathcal{K}_M \subset H \quad \text{and} \quad \partial H \text{ is transverse to } F.$$

So F' is also reflective with corresponding H'. Then there is $\delta > 0$, and a neighborhood of ∂H which is diffeomorphic to $\partial H \times [-\delta, \delta]$, and so that F restricted to $\partial H \times [-\delta, \delta]$ has leaves of the form $(L \cap \partial H) \times [-\delta, \delta]$. We may assume that the foliation preserving diffeomorphism φ extends to $\partial H \times [-\delta, \delta]$, and that $\varphi(\partial H \times [-\delta, \delta])$ is diffeomorphic to $\partial H' \times [-\delta, \delta]$, and that it has the same properties as $\partial H \times [-\delta, \delta]$. Then we may form the compact foliated manifold

$$\widehat{M} \;=\; H \cup_{\widehat{\varphi}} H',$$

where $\widehat{\varphi} : \partial H \times [-\delta, \delta] \to \partial H' \times [-\delta, \delta]$ is given by $\widehat{\varphi}(x, s) = (\varphi(x), -s)$. We change the orientation of F' to the opposite of what it was originally. The resulting foliation $F \cup_{\widehat{\varphi}} F'$ is denoted $\widehat{F}$. Denote by $\pi : \partial H \times [-\delta, \delta] \to \partial H$ the projection and note that $E\,|_{\partial H \times [-\delta, \delta]} \simeq \pi^*(E\,|_{\partial H})$, and $TF\,|_{\partial H \times [-\delta, \delta]} \simeq \pi^*(TF\,|_{\partial H})$. (Note that $\dim(TF\,|_{\partial H}) = \dim(TF)$, not $\dim(TF) - 1 = \dim F\,|_{\partial H}$.) We may assume that ∇ and D_F are preserved under the maps $(x, s) \to (x, -s)$ and $E_{(x,s)} \to E_{(x,-s)}$. This implies that D_F and $D'_{F'}$ are identified under the gluing map. In addition, ω and ω' fit together, giving $\widehat{\omega}$. This construction is the exact translation of the Gromov-Lawson construction to foliations.

Finally, denote the leafwise operator on $\widehat{F}$ by $\widehat{D}_{\widehat{F}}$ (and its associated projections by $\widehat{P}_0$ and $\widehat{P}_{(0,\epsilon)}$). Then we have the following extension of Alain Connes' celebrated index theorem, see [13], which is very useful in Section 7.

Theorem 6.7. *With the above notations, suppose that F (and so also F') is reflective, but not necessarily Riemannian. Suppose further that (C, C') is a compatible near infinity pair of closed holonomy invariant currents, with associated current $\widehat{C}$. Then*

$$\langle \mathrm{ch}(\mathrm{Ind}_a(D, D'), (C, C')\rangle \;=\; \langle \mathrm{ch}(\mathrm{Ind}_a(\widehat{D}_{\widehat{F}})), \widehat{C}\rangle.$$

Proof. We prove the case where $(C, C') = [\omega_T, \omega'_{T'}]$, since it is notationally simpler. The general case is left to the reader. Theorems 4.2 and 5.5 give

$$\langle \mathrm{ch}(\mathrm{Ind}_a(D, D')), [\omega_T, \omega'_{T'}]\rangle \;=\; \int\limits_{\mathcal{K}_M} \mathrm{AS}(D_F) \wedge \omega - \int\limits_{\mathcal{K}'_{M'}} \mathrm{AS}(D'_{F'}) \wedge \omega',$$

since the differential forms $\mathrm{AS}(D_F) \wedge \omega$ and $\mathrm{AS}(D'_{F'}) \wedge \omega'$ are φ compatible off $\mathcal{K}_M$ and $\mathcal{K}'_{M'}$. Next,

$$\int\limits_{\mathcal{K}_M} \mathrm{AS}(D_F) \wedge \omega - \int\limits_{\mathcal{K}'_{M'}} \mathrm{AS}(D'_{F'}) \wedge \omega' \;=\; \int\limits_{H} \mathrm{AS}(D_F) \wedge \omega - \int\limits_{H'} \mathrm{AS}(D'_{F'}) \wedge \omega' \;=$$

$$\int\limits_{\widehat{M}} \mathrm{AS}(\widehat{D}_{\widehat{F}}) \wedge \widehat{\omega} \;=\; \langle \int\limits_{\widehat{F}} \mathrm{AS}(\widehat{D}_{\widehat{F}}), \widehat{\omega}_{\widehat{T}}\rangle \;=\; \langle \mathrm{ch}(\mathrm{Ind}_a(\widehat{D}_{\widehat{F}})), \widehat{\omega}_{\widehat{T}}\rangle.$$

The last equality is from Theorem 6.2 of [4] applied to the closed foliated manifold $(\widehat{M}, \widehat{F})$. The others are obvious. □

Note that, since the integrands $\mathrm{AS}(D_F) \wedge \omega$ and $\mathrm{AS}(D'_{F'}) \wedge \omega'$ are φ compatible off $\mathcal{K}_M$ and $\mathcal{K}'_{M'}$, this result is actually independent of the choice of the transverse compact hypersurface ∂H and for simplicity we may assume that $H = \mathcal{K}_M$.

Theorem 6.8. *Suppose that (M, F, D), (M', F', D') are as in Theorem 6.7. Suppose furthermore that $\widehat{F}$ is Riemannian, that $\widehat{P}_0$ and $\widehat{P}_{(0,\epsilon)}$ are transversely smooth, and the Novikov-Shubin invariants of $\widehat{D}_{\widehat{F}}$ are greater than $\ell/2$, for some $0 \leq \ell \leq q$. Then for any ℓ homogeneous φ-compatible pair (ω, ω') as before,*

$$\langle \mathrm{ch}(\mathrm{Ind}_a(D, D')), [\omega_T, \omega'_{T'}]\rangle = \langle (\mathrm{ch}(\widehat{P}_0), \widehat{\omega}_{\widehat{T}}\rangle.$$

Moreover, if we impose on (M, F, D) and (M', F', D') the assumptions of Theorem 4.6, then we have

$$\langle (\mathrm{ch}(P_0), \mathrm{ch}(P_0)), (\omega_T, \omega'_{T'})\rangle \;=\; \langle (\mathrm{ch}(\widehat{P}_0), \widehat{\omega}_{\widehat{T}}\rangle.$$

This is a consequence of Theorem 6.7 using Theorem 4.1 of [5] to deduce the second equality, with $\langle (\mathrm{ch}(\widehat{P}_0), \widehat{\omega}_{\widehat{T}}\rangle$ being well defined under our assumptions.

Remark 6.9. This result raises some interesting questions.

1. Suppose that $\mathcal{R}_F$, so also $\mathcal{R}_{F'}$, is strictly positive near infinity, then $\langle \mathrm{ch}(P_0), \omega_T \rangle$ and $\langle \mathrm{ch}(P_0')), \omega'_{T'} \rangle$ exist. Under what more general conditions than those in Theorems 4.6 and 6.8 does

$$\langle \mathrm{ch}(P_0), \omega_T \rangle - \langle \mathrm{ch}(P_0'), \omega'_{T'} \rangle = \langle \mathrm{ch}(\widehat{P}_0), \widehat{\omega}_{\widehat{T}} \rangle?$$

2. In general, suppose that $\langle \mathrm{ch}(P_0), \omega_T \rangle - \langle \mathrm{ch}(P_0'), \omega'_{T'} \rangle - \langle \mathrm{ch}(\widehat{P}_0), \widehat{\omega}_{\widehat{T}} \rangle \neq 0$. What can be said about the geometry or topology of (M, F, D), (M', F', D'), and $(\widehat{M}, \widehat{F}, \widehat{D})$?
3. How are the Novikov-Shubin invariants of D and D' related to those of $\widehat{D}$?

The previous construction extends to the following more general situation to yield the so called higher Φ relative index theorem, see again [20]. In particular, we assume that (M, F) and (M', F') satisfy the hypotheses of Theorem 4.6, with the following changes. In particular, $M \smallsetminus \mathcal{K} = V_+ \cup V_\Phi$ and $M' \smallsetminus \mathcal{K}' = V'_+ \cup V'_\Phi$, where the unions are disjoint. For this case, $\Phi = (\phi, \varphi)$ is a bundle morphism from $E \to V_\Phi$ to $E' \to V'_\Phi$ as in Section 2, our good covers $\mathcal{U}$ and $\mathcal{U}'$ are compatible on V_Φ and V'_Φ, and ω and ω' are φ compatible on V_Φ and V'_Φ. We assume that F is transverse to ∂V_Φ, so F' is transverse to $\partial V'_\Phi$. Finally, we assume that $\mathcal{R}_F^E$ and $\mathcal{R}_{F'}^{E'}$ are strictly positive off $\mathcal{K}$ and $\mathcal{K}'$, so we do not need the assumptions on the integrals being finite.

Next, consider as above the manifold $\widehat{M} = (M \smallsetminus V_\Phi) \cup_{\widehat{\varphi}} (M' \smallsetminus V'_\Phi)$, with the foliation

$$\widehat{F} \;=\; (F|_{M \smallsetminus V_\Phi}) \cup_{\widehat{\varphi}} (F'|_{M' \smallsetminus V'_\Phi}),$$

where the orientation on $\widehat{F}|_{M \smallsetminus V_\Phi}$ is the one on F, and that on $\widehat{F}|_{M' \smallsetminus V'_\Phi}$ is the opposite of the one on F'. We also have the bundle $\widehat{E} \to \widehat{M}$ induced by E and E', the leafwise operator $\widehat{D}_{\widehat{F}}$ induced by D_F and $D'_{F'}$, and the differential form $\widehat{\omega}$ induced by ω and ω'.

Because of the positivity off compact subsets, all three operators have finite indices, and we have the following.

Theorem 6.10. *[The higher foliated Φ-index formula]*

$$\langle \mathrm{ch}(\mathrm{Ind}_a(\widehat{D})), \widehat{\omega}_{\widehat{T}} \rangle \;=\; \langle \mathrm{ch}(\mathrm{Ind}_a(D)), \omega_T \rangle \;-\; \langle \mathrm{ch}(\mathrm{Ind}_a(D')), \omega'_{T'} \rangle.$$

The proof follows from our results here, by easily adapting the proof of Theorem 4.35 of [20].

7. Applications

7.1. Leafwise PSC and the higher Gromov-Lawson invariant

We further extend the Gromov-Lawson construction in [20], Section 3, see also [27], IV.7, to get an invariant for the space of PSC metrics on a foliation F whose tangent bundle admits a spin structure. We calculate this invariant for a large collection of spin foliations whose Haefliger $\widehat{A}$ genus is zero, so the results of [8] do not apply. Using the higher index results here, we show that the space of PSC metrics on each of these foliations has infinitely many path connected components, thus verifying our claims that higher order index theorems allow for the extension of results for manifolds with non-zero $\widehat{A}$ genus to arbitrary manifolds, and that the higher order terms of the $\widehat{A}$ genus also carry geometric information.

For simplicity, we assume that M is compact. Denote by $\mathcal{M}$ the space of all smooth metrics on F with the C^∞ topology, and by $\mathcal{M}^+_{sc} \subset \mathcal{M}$ the subspace of metrics with PSC along the leaves.

Scalar curvature and the so called Atiyah-Singer operator are intimately related. Recall that $\mathcal{S}_F$ is the bundle of spinors along the leaves of F, with the leafwise spin connection ∇^F. The leafwise Atiyah-Singer operator is the leafwise spin Dirac operator $D^{\mathcal{S}}_F = (D^{\mathcal{S}}_L)$, which acts on $\mathcal{S}_F$, as usual, by

$$D^{\mathcal{S}}_L \;=\; \sum_{i=1}^{p} \rho(X_i)\nabla^F_{X_i},$$

where $X_1, \ldots, X_p$ is a local oriented orthonormal basis of TL, and $\rho(X_i)$ is the Clifford action of X_i on the bundle $\mathcal{S}_F|L$. Denote by κ the leafwise scalar curvature of F, that is

$$\kappa = -\sum_{i,j=1}^{p} \langle R_{X_i,X_j}(X_i), X_j\rangle,$$

where R is the curvature operator associated to the metric on the leaves of F. In this case the Bochner Identity, Equation 6.1, is quite simple, see [27], namely

$$(D^{\mathcal{S}}_L)^2 = (\nabla^F)^*\nabla^F \;+\; \frac{1}{4}\kappa. \tag{7.1}$$

Consider the foliation $F_{\mathbb{R}}$ on $M_{\mathbb{R}} = M \times \mathbb{R}$ with leaves $L_{\mathbb{R}} = L \times \mathbb{R}$ and with the leafwise volume form $dx_F \times dt$. If $\mathcal{U}$ is a good cover of M, $\mathcal{U}_{\mathbb{R}} = \{(U^n_i, T^n_i) = (U_i \times (3n-2, 3n+2), T_i) |\, (U_i, T_i) \in \mathcal{U}, n \in \mathbb{Z}\}$ is a good cover of $M_{\mathbb{R}}$. Denote by $\pi : M_{\mathbb{R}} \to M$ the projection. Suppose that $g_0, g_1 \in \mathcal{M}^+_{sc}$, and $(g_t)_{t\in[0,1]}$ is a smooth family in $\mathcal{M}$ from g_0 to g_1. On $F_{\mathbb{R}}$, set $G = g_0 + dt^2$ for $t \leq 0$, $G = g_1 + dt^2$ for $t \geq 1$, and $G = g_t + dt^2$ for $0 < t < 1$.

The leafwise spin Dirac operator $D_F^{\mathcal{S}}$ extends to the leafwise spin Dirac operator $D_{\mathbb{R}}$ on $F_{\mathbb{R}}$. Following Gromov-Lawson, [20], Equation (3.13), we set

$$i(g_0, g_1) = \mathrm{ch}(\mathrm{Ind}_a(D_{\mathbb{R}})) \ \in \ H_c^*(M_{\mathbb{R}}/F_{\mathbb{R}}). \tag{7.2}$$

Theorem 7.3. *$i(g_0, g_1)$ depends only on g_0 and g_1. If $i(g_0, g_1) \neq 0$, then g_0 and g_1 are not in the same path connected component of $\mathcal{M}_{sc}^+$.*

Proof. Suppose that g_t and $\widehat{g}_t$ are two smooth families of metrics in $\mathcal{M}$ from g_0 to g_1, with associated metrics G and $\widehat{G}$ and associated operators $D_{\mathbb{R}}$ and $\widehat{D}_{\mathbb{R}}$. A byproduct of Theorem 4.2 is that $i(g_0, g_1) \ = \ \left[\int_{F_{\mathbb{R}}} \widehat{A}(TF_{\mathbb{R}}, G)\right]$, where $\widehat{A}(TF_{\mathbb{R}}, G)$ is the Atiyah-Singer characteristic differential form, the so-called A-hat form of F, on $M_{\mathbb{R}}$ associated to the metric G, and similarly for $\widehat{G}$. Thus we have

$$\begin{aligned} i(g_0, g_1)(G) - i(g_0, g_1)(\widehat{G}) &= \left[\int_{F_{\mathbb{R}}} (\widehat{A}(TF_{\mathbb{R}}, G) - \widehat{A}(TF_{\mathbb{R}}, \widehat{G}))\right] \\ &= \left[\int_{F_{\mathbb{R}}^0} (\widehat{A}(TF_{\mathbb{R}}, G) - \widehat{A}(TF_{\mathbb{R}}, \widehat{G}))\right], \end{aligned}$$

where $F_{\mathbb{R}}^0$ is the foliation on $\bigcup_i U_i^0$. The forms $\widehat{A}(TF_{\mathbb{R}}, G)$ and $\widehat{A}(TF_{\mathbb{R}}, \widehat{G})$ are locally computable in terms of their associated curvatures. Thus, off the compact subset $M \times [0, 1]$, they agree, which justifies the second equality. By abuse of notation, we may write

$$\int_{F_{\mathbb{R}}^0} (\widehat{A}(TF_{\mathbb{R}}, G) - \widehat{A}(TF_{\mathbb{R}}, \widehat{G})) \ = \ \int_{F \times [0,1]} (\widehat{A}(TF_{\mathbb{R}}, G) - \widehat{A}(TF_{\mathbb{R}}, \widehat{G})).$$

Since the cohomology classes of the two forms are the same, $\widehat{A}(TF_{\mathbb{R}}, G) - \widehat{A}(TF_{\mathbb{R}}, \widehat{G})$ is an exact form $d_{M \times \mathbb{R}}\Psi$, which is locally computable in terms of the curvatures and connections. In particular, $\Psi = 0$ on the closure of open sets where the connections agree. So off $M \times (0, 1)$, Ψ is zero, since the connections agree there. Thus

$$\int_{F_{\mathbb{R}}^0} (\widehat{A}(TF_{\mathbb{R}}, G) - \widehat{A}(TF_{\mathbb{R}}, \widehat{G})) \ = \ \int_{F \times [0,1]} d_{M \times \mathbb{R}}\Psi \ = \ d_H \int_{F \times [0,1]} \Psi,$$

and $i(g_0, g_1)(G) \ - \ i(g_0, g_1)(\widehat{G}) = 0$ in $H_c^*(M_{\mathbb{R}}/F_{\mathbb{R}})$.

For the second part, assume that g_0 and g_1 are in the same path connected component of $\mathcal{M}_{sc}^+$, and that g_t, is a smooth family of metrics in $\mathcal{M}_{sc}^+$ from g_0 to g_1. Then, after replacing g_t with a suitable reparametrization, a so-called "warped product" metric, G

restricted to each leaf of $F_{\mathbb{R}}$ will have PSC, see [19], p.184, and since the family of metrics is smooth, it is strictly positive. Then, Proposition 6.3 gives that $P^{\mathbb{R}}_{[0,\epsilon]} = 0$, for some positive ϵ, so the Novikov -Shubin invariants are infinite and Remark 4.4 (1) gives that $i(g_0, g_1) = 0$. □

Remark 7.4. Theorem 7.3 remains true if we consider concordance classes of PSC metrics, which a priori is stronger. Recall that leafwise metrics are concordant if there is a metric G on $TF_{\mathbb{R}}$ so that it agrees with $g_0 \times dt^2$ near $-\infty$ and with $g_1 \times dt^2$ near $+\infty$. The conclusion is that if $i(g_0, g_1) \neq 0$, then g_0 and g_1 are not in the same concordance class of metrics in $\mathcal{M}^+_{sc}$. The proof being essentially the same.

Remark 7.5. We could also extend this theory to concordance classes of leafwise flat connections ∇ on an auxiliary bundle E. The invariant would become $i((g_0, \nabla_0), (g_1, \nabla_1))$. See [3]. The theorem would then be that if g_0 and g_1 are concordant, and ∇_0 and ∇_1 can be joined by leafwise flat connections, then $i((g_0, \nabla_0), (g_1, \nabla_1)) = 0$.

Next, we have a corollary of Theorem 6.10.

Corollary 7.6. *Suppose* $g_0, g_1, g_2 \in \mathcal{M}^+_{sc}$. *Then*

$$i(g_0, g_1) + i(g_1, g_2) \;=\; i(g_0, g_2), \quad \textit{so}\;\; i(g_0, g_1) + i(g_1, g_2) + i(g_2, g_0) \;=\; 0.$$

Proof. In the notation of Theorem 6.10, take (M, F), (M', F') and $(\widehat{M}, \widehat{F})$ to be $(M_{\mathbb{R}}, F_{\mathbb{R}})$, $\mathcal{K} = \mathcal{K}' = M \times [0, 1]$, $V_\Phi = V'_\Phi = M \times (-\infty, 0)$, and $V_+ = V'_+ = M \times (1, \infty)$. To compute $i(g_i, g_j)$ take

$$G_{i,j} = g_i + dt^2 \text{ for } t \in (-\infty, 0], \text{ and } G_{i,j} = g_j + dt^2 \text{ for } t \in [1, \infty).$$

For the first, we have

$$i(g_0, g_1) - i(g_0, g_2) = \mathrm{ch}(\mathrm{Ind}_a(D_{\mathbb{R}}(G_{0,1}))) - \mathrm{ch}(\mathrm{Ind}_a(D_{\mathbb{R}}(G_{0,2}))) =$$
$$\mathrm{ch}(\mathrm{Ind}_a(D_{\mathbb{R}}(G_{2,1})) = i(g_2, g_1) = -i(g_1, g_2).$$

The second equality is from Theorem 6.10, where $\widehat{D}^{\widehat{E}}_L = D_{\mathbb{R}}(G_{2,1})$, $D^E_L = D_{\mathbb{R}}(G_{0,1})$, and $D^{E'}_L = D_{\mathbb{R}}(G_{0,2})$.

The second equation is then obvious, as $i(g_0, g_2) = -i(g_2, g_0)$. □

Now suppose that M is the boundary of a compact manifold W with a spin foliation $\widehat{F}$ which is transverse to M, and which restricts to F there. Extend $\widehat{F}$ as above to $W \cup_M (M \times [0, \infty))$. Given a metric g of PSC on F, extend it to a complete leafwise metric $\widehat{g}$ on $\widehat{F}$ by making it $g + dt^2$ on $M \times [-\epsilon, \infty)$, where $M \times [-\epsilon, 0]$ is a collar neighborhood of $M \subset W$, and extending it arbitrarily over the rest of the interior of W.

Definition 7.7. $i(g, W) = \mathrm{ch}(\mathrm{Ind}_a(\widehat{D}_{\widehat{F}}))$.

Note that Theorem 4.2 and the proof of Theorem 7.3, show that $i(g, W)$ does not depend on the extension of g over W. It does however depend on W in general.

In this situation, we have the following two corollaries of Theorem 4.2.

Corollary 7.8. *Suppose that $g_0, g_1 \in \mathcal{M}^+_{sc}$. Then*

$$i(g_0, g_1) = i(g_1, W) - i(g_0, W),$$

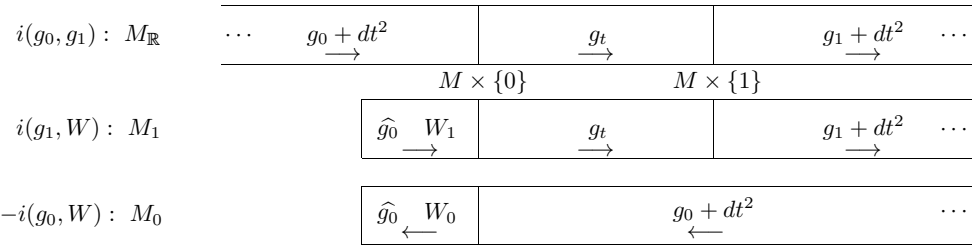
as Haefliger classes. In addition, if $\widehat{g}_0$ has PSC, then $i(g_0, W) = 0$.

The reader may wonder how the classes in the first equality can be compared, since they are on different manifolds. This is explained below.

Proof. Consider the following

- $(M_{\mathbb{R}}, F_{\mathbb{R}})$ with the metric $G_{0,1}$ above, giving $i(g_0, g_1)$.
- $M_0 = W_0 \cup_M (M \times [0, \infty))$ with the metric $g_0 + dt^2$ on $M \times [0, \infty)$, and the metric $\widehat{g}_0$ on $W_0 = W$. Take the opposite orientation on M_0 by reversing the orientations on $[0, \infty)$ and W_0, so this gives $-i(g_0, W)$.
- $M_1 = W_1 \cup_M (M \times [0, \infty))$ with the metric $G_{0,1}$ restricted to $M \times [0, \infty)$, and the metric $\widehat{g}_0$ on $W_1 = W$. As the metric on $M \times [1, \infty)$ is $g_1 \times dt^2$, this gives $i(g_1, W)$.

The meaning of the first equality is that representatives of the classes on $M_0 \dot{\cup} M_1 \setminus W_0 \dot{\cup} W_1$ equal the representative on $M_{\mathbb{R}}$, while what remains on W_0 and W_1 cancel. It is useful to have a picture of the situation. The arrows indicate the orientations.



We may use the $\widehat{A}$-forms associated to the terms, since they are arbitrarily close to differential forms in the Haefliger classes. We indicate them by $\widehat{A}(M_{\mathbb{R}})$, $\widehat{A}(M_0)$, and $\widehat{A}(M_1)$. Then,

- $\widehat{A}(M_1)$ restricted to $M_1 \setminus W_1$ equals $\widehat{A}(M_{\mathbb{R}})$ restricted to $M \times (0, \infty)$;
- $\widehat{A}(M_0)$ restricted to $M_0 \setminus W_0$ equals $\widehat{A}(M_{\mathbb{R}})$ restricted to $M \times (-\infty, 0]$;
- $\widehat{A}(M_0)$ restricted to W_0 cancels $\widehat{A}(M_1)$ restricted to W_1.

For the second statement, Proposition 6.3 gives that there is $\epsilon > 0$ so that $P_{[0,\epsilon]} = 0$. Then Theorem 4.5 gives $i(g_0, W) = 0$. □

Corollary 7.9. *Suppose that $\widehat{g}_0$ has PSC, and that g_1 extends to $\widehat{g}_1$ with PSC over a compact manifold $\widehat{W}_1$ with the spin foliation $\widehat{F}_1$ extending F. Set*

$$X_{(0,1)} \;=\; W \cup_M (M \times [0,1]) \cup_M \widehat{W}_1$$

with the metric $\widehat{G}_{(0,1)}$ which is $\widehat{g}_0$ on W, $G_{0,1}$ on $M \times [0,1]$ and $\widehat{g}_1$ on $\widehat{W}_1$. Denote the leafwise operator on the foliation $F_{(0,1)}$ of $X_{(0,1)}$ by $D_{(0,1)}$. Then

$$i(g_0, g_1) \;=\; \mathrm{ch}(\mathrm{Ind}_a(D_{(0,1)})) \;=\; \int\limits_{F_{(0,1)}} \widehat{A}(TF_{(0,1)}).$$

Proof. For $i(g_0, g_1) = \mathrm{ch}(\mathrm{Ind}_a(D_{(0,1)}))$, set $\widehat{M}_1 = \widehat{W}_1 \cup_M (M \times [0,\infty))$ with the metric $g_1 + dt^2$ on $M \times [0,\infty)$, and the metric $\widehat{g}_1$ on $\widehat{W}_1$, so the metric has PSC everywhere and $i(g_1, \widehat{W}_1) = 0$. Then, we have,

$$i(g_0, g_1) = i(g_1, W) - i(g_0, W)) = i(g_1, W) = i(g_1, W) - i(g_1, \widehat{W}_1) = \mathrm{ch}(\mathrm{Ind}_a(D_{(0,1)})).$$

The first three equalities are obvious. For the last, proceed as in the first part, noting that

- $\widehat{A}(M_1)$ restricted to $W_1 \cup_M (M \times [0,1])$ equals $\widehat{A}(X_{(0,1)})$ restricted to $W \cup_M (M \times [0,1])$;
- -$\widehat{A}(\widehat{M}_1)$ restricted to $\widehat{W}_1$ cancels $\widehat{A}(X_{(0,1)})$ restricted to $\widehat{W}_1$;
- -$\widehat{A}(\widehat{M}_1)$ restricted to $\widehat{M}_1 \setminus \widehat{W}_1$ cancels $\widehat{A}(M_1)$ restricted to $M_1 \setminus (M_1 \times (1,\infty))$.

Finally, the fact that $\mathrm{ch}(\mathrm{Ind}_a(D_{(0,1)})) = \int\limits_{F_{(0,1)}} \widehat{A}(TF_{(0,1)})$ is a result from [4]. □

7.2. Some examples

To finish, we construct a large collection of spin foliations whose space of leafwise PSC metrics has infinitely many path connected components.

Suppose we have the following data.

- A closed foliated manifold (M, F), with F spin and $\int\limits_F \widehat{A}(TF) \neq 0$ in $H^*_c(M/F)$.
- A closed manifold S and a family (g_i) of PSC metrics on it, and compact spin manifolds X_i with boundary S and metric $\widehat{g}_i$, which is $g_i \times dt^2$ in a neighborhood of S, and $\widehat{g}_i$ also has PSC. Set

$$X_{(i,j)} = X_i \cup (S \times [0,1]) \cup X_j,$$

where the metric on $S \times [0,1]$ is $g_t \times dt^2$, and g_t is a path of metrics from g_i to g_j. Assume further that $i(g_i, g_j)$ is non-zero.

Proposition 7.10. *The foliated manifold $(M \times S, TF \times TS)$ has a family of PSC metrics $(\mathfrak{g}_i)$, so that for any $i \neq j$, $\mathfrak{g}_i$ and $\mathfrak{g}_j$ do not belong to the same path component of the space of PSC metrics on $TF \times TS$.*

Proof. Since M is compact, F admits a metric g of bounded scalar curvature. Set $\mathfrak{g}_i = g \times c_i g_i$, where $c_i \in (0, \infty)$ is such that $\mathfrak{g}_i$ has PSC. For the manifold $M \times X_{(i,j)}$, with the foliation $F \times X_{(i,j)}$, Corollary 7.9 gives

$$i(\mathfrak{g}_i, \mathfrak{g}_j) = \int\limits_{F \times X_{(i,j)}} \widehat{A}(TF \times TX_{(i,j)}).$$

If $\mathfrak{g}_i$ and $\mathfrak{g}_j$ were in the same path component of the space of PSC metrics on $TF \times TS$, then we would have $i(\mathfrak{g}_i, \mathfrak{g}_j) = 0$. However, if $i \neq j$, then

$$\int\limits_{F \times X_{(i,j)}} \widehat{A}(TF \times TX_{(i,j)}) = \int\limits_{F \times X_{(i,j)}} \widehat{A}(TF)\widehat{A}(TX_{(i,j)}) = \int\limits_F \widehat{A}(TF) \int\limits_{X_{(i,j)}} \widehat{A}(TX_{(i,j)})$$

$$= i(\mathfrak{g}_i, \mathfrak{g}_j) \int\limits_F \widehat{A}(TF) \neq 0. \quad \square$$

Here are examples of this type.

Example 7.11. We adapt Example 1 of [22]. In particular, let $G = SL_2\mathbb{R} \times \cdots \times SL_2\mathbb{R}$ (q copies) and $K = SO_2 \times \cdots \times SO_2$ (q copies). G acts naturally on $\mathbb{R}^{2q} \smallsetminus \{0\}$ and is well known to contain subgroups Γ with $N = \Gamma\backslash G/K$ compact, (in fact a product of q surfaces of higher genus). Set

$$M = \Gamma\backslash G \times_K ((\mathbb{R}^{2q} \smallsetminus \{0\})/\mathbb{Z}) \simeq \Gamma\backslash G \times_K (\mathbb{S}^{2q-1} \times \mathbb{S}^1),$$

where $n \in \mathbb{Z}$ acts on $\mathbb{R}^{2q} \smallsetminus \{0\}$ by $n \cdot z = e^n z$.

M has two transverse foliations, F which is given by the fibers $\mathbb{S}^{2q-1} \times \mathbb{S}^1$ of the fibration $M \to N$, and a transverse foliation coming from the foliation τ of Example 1 of [22]. More precisely, τ is defined on the vector bundle $\Gamma\backslash G \times_K \mathbb{R}^{2q}$, and the zero section is a leaf of it. In addition, the action of $\mathbb{Z}$ preserves τ, fixing the zero section, so it descends to a foliation on M, also denoted τ.

We work with F, noting that TF is orientable and spin since $\mathbb{R}^{2q} - \{0\}$ has these structures and the actions of K and $\mathbb{Z}$ preserve them. It also happens to admit a met-

ric with PSC, namely the product of the standard metrics on $\mathbb{S}^{2q-1}$ and $\mathbb{S}^1$, which is preserved by the action of K. The following proposition is proven in the appendix.

Proposition 7.12. *$\int_F \widehat{A}(TF)$ is a nowhere zero $2q$ form on N. In particular, there is a non-zero constant C_q so that $\int_N \int_F \widehat{A}(TF) = C_q \operatorname{vol}(N)$.*

Thus, $\int_F \widehat{A}(TF) \neq 0$ in $H_c^*(M/F)$. Note that this also shows that the Haefliger $\widehat{A}$ genus of TF, i.e. $\left[\int_F \widehat{A}(TF)\right]^0 \in H_c^0(M/F)$, is zero, which is why we cannot use the results of [8] here.

In [11], Carr constructs examples of "exotic" PSC metrics $\mathfrak{g}_i$, $i \in \mathbb{Z}_+$ on $\mathbb{S}^{4k-1}$, for $k > 1$, and compact Riemannian $4k$ dimensional spin manifolds X_i with boundary $\mathbb{S}^{4k-1}$, so that the metric $\widehat{\mathfrak{g}}_i$ on X_i is $\mathfrak{g}_i \times dt^2$ in a neighborhood of $\mathbb{S}^{4k-1}$, and $\widehat{\mathfrak{g}}_i$ also has PSC. Set

$$X_{(i,j)} \;=\; X_i \cup (\mathbb{S}^{4k-1} \times [0,1]) \cup X_j,$$

where the metric on $\mathbb{S}^{4k-1} \times [0,1]$ is $\mathfrak{g}_t \times dt^2$, and $\mathfrak{g}_t$ is a path of metrics from $\mathfrak{g}_i$ to $\mathfrak{g}_j$. These examples have the property that $i(\mathfrak{g}_i, \mathfrak{g}_j) \neq 0$. Thus we have all the elements required to apply Proposition 7.10

Remark 7.13. Note that the calculations in the examples in [22] can be used to provide examples associated to the groups $G = SL_{2n_1}\mathbb{R} \times \cdots \times SL_{2n_r}\mathbb{R}$, and $K = SO_{2n_1} \times \cdots \times SO_{2n_r}$, and $G = SL_{2n_1}\mathbb{R} \times \cdots \times SL_{2n_r}\mathbb{R} \times \mathbb{R}$ and $K = SO_{2n_1} \times \cdots \times SO_{2n_r} \times \mathbb{Z}$. We leave the details and further extensions to the reader.

The next example is an easy corollary of the Kreck-Stolz result from [26][Corollary 2.15].

Proposition 7.14. *Suppose that (M, F) is a closed foliated manifold with F spin. Let Y be a closed connected spin manifold of dimension $4k-1 > 3$ with vanishing real Pontrjagin classes and such that $H^1(Y; \mathbb{Z}/2) = 0$. If Y admits a PSC metric, then the foliated manifold $(M \times Y, TF \times TY)$ admits a sequence $(\mathfrak{g}_i)$ of leafwise PSC metrics such that for any $i \neq j$, $\mathfrak{g}_i$ and $\mathfrak{g}_j$ are not in the same path component of PSC metrics on $TF \times TY$.*

Notice that if Y is for instance simply connected, then it always admits a metric of PSC by [31].

Proof. In [26], Kreck and Stolz produce an infinite sequence g_i of PSC metrics on Y such that for any $i \neq j$, the Gromov-Lawson invariant $i_{GL}(g_i, g_j) \neq 0$. Note that $i_{GL}(g_i, g_j)$ is the difference of the dimensions of the positive and negative parts of the kernel of $D_{\mathbb{R}}$ on the manifold $Y_{\mathbb{R}}$. Thus, there is a non-trivial L^2 element ζ in the kernel of $D_{\mathbb{R}}$. On the foliated manifold $(M \times Y, TF \times TY)$ there is the sequence of PSC metrics $(\mathfrak{g}_i)$, where $\mathfrak{g}_i$ is as in Proposition 7.10. For $i \neq j$, these metrics are not in the same path component of leafwise PSC metrics. For if they were, then the foliation $TF \times TY_{\mathbb{R}}$, would have PSC everywhere. So, by Proposition 6.3, there would not be any non-trivial L^2 elements in the kernel of $D_{\mathbb{R}}$. But this is patently false as $(0, \zeta)$ is such a non-trivial L^2 element. □

We finish with some cogent comments by the referee. It would be interesting to know if, in any of the examples above, the restrictions of the $\mathfrak{g}_i$ to each leaf are pairwise isotopic. This would rule out the possibility of showing non-isotopy by restricting to a single leaf and applying results from the non-foliation case. A particularly interesting case is that when $q = 2$ in Example 7.11 where the fibers are $\mathbb{S}^3 \times \mathbb{S}^1$. Very little is known about the topology of PSC metrics on 4-manifolds. It is an open question whether the space of PSC metrics on $\mathbb{S}^4$ is path connected and the status is likely the same on $\mathbb{S}^3 \times \mathbb{S}^1$.

Appendix A. Proof of Proposition 7.12

We follow the proof of Theorem 5.4 in [22]. Denote by $(x_1, y_1, .., x_q, y_q)$ the coordinates on $\mathbb{R}^{2q}$. Choose nonzero numbers $\lambda_1, ...\lambda_q \in \mathbb{R}$, and set

$$X_\lambda = \sum_{i=1}^{q} \lambda_i (x_i \partial/\partial x_i + y_i \partial/\ \partial y_i).$$

This vector field has an isolated singularity at the origin and it commutes with the actions of K and $\mathbb{Z}$ on $\mathbb{R}^{2q} \smallsetminus \{0\}$. Thus it induces a nowhere zero vector field also denoted X_λ on the bundle M.

Denote by ω_λ the one-form on $\mathbb{R}^{2q} \smallsetminus \{0\}$ defined by

$$\omega_\lambda = \sum_{i=1}^{q} \frac{\lambda_i}{x_i^2 + y_i^2} (x_i dx_i + y_i dy_i).$$

Note carefully that this is different from the ω_λ of [22]. This change is necessary so that ω_λ is invariant under the action of $\mathbb{Z}$. Note also that $d\omega_\lambda = 0$ still holds. The actions of K and $\mathbb{Z}$ on $\mathbb{R}^{2q} \smallsetminus \{0\}$ preserve ω_λ, so it induces a one-form ω_λ on M.

Let S be the sphere bundle in $\widehat{M} = \Gamma \backslash G \times_K ((\mathbb{R}^{2q} \smallsetminus \{0\}))$ given by the image of

$$\{(g, (x_1, y_1, ..., x_q, y_q)) \in G \times (\mathbb{R}^{2q} \smallsetminus \{0\}) \mid \sum_{i=1}^{q} \lambda_i (x_i^2 + y_i^2) = 1\}.$$

S is invariant under Γ and K so it is well defined. Set

$$S_0 = S = 0 \cdot S, \quad \text{and} \quad S_1 = 1 \cdot S.$$

Note that the condition on S_1 is $\sum_{i=1}^{q} \lambda_i (x_i^2 + y_i^2) = e^2$, so its radius is e. Then we may write,

$$M = \Gamma \backslash G \times_K (\mathbb{S}^{2q-1} \times [1, e]),$$

where we identify the boundary components, S_0 and S_1, on the right, and we may do our computations, as in [22], using the coordinates on $G \times \mathbb{S}^{2q-1} \times (1, e)$.

Denote by θ the unique basic connection (for the foliation τ!) on $T(\mathbb{S}^{2q-1} \times (1, e))$, which is the normal bundle of τ, whose covariant derivative ∇ satisfies, for all $Y \in T(\mathbb{S}^{2q-1} \times (1, e))$,

$$\nabla_Y \partial/\partial x_i = \omega_\lambda(Y)[X_\lambda, \partial/\partial x_i], \text{ and } \nabla_Y \partial/\partial y_i = \omega_\lambda(Y)[X_\lambda, \partial/\partial y_i].$$

The proof in [22] works just as well here to show that θ is well defined.

The computation of the curvature Ω of ∇ proceeds just as in [22]. In particular, we may assume that we have a neighborhood U in N whose inverse image in M is of the form $U \times (\mathbb{S}^{2q-1} \times (1, e))$, and coordinates on it, so that the local form of Ω with respect to the local basis $\partial/\partial x_1, \partial/\partial y_1, ...\partial/\partial x_q, \partial/\partial y_q$ of $T(\mathbb{S}^{2q-1} \times (1, e))$, is given by

$$\Omega_{2i-1}^{2i-1} = \Omega_{2i}^{2i} = \lambda_i d(\lambda\delta),$$

and all other terms are zero. Here, for $i = 1, ..., q$,

- $\lambda\delta = \lambda_1\delta_1 + \cdots + \lambda_q\delta_q$;
- $\delta_i = x_i y_i \omega_i + \frac{1}{2}(x_i^2 - y_i^2)\gamma_i$;
- $\omega_1, \gamma_1, ..., \omega_q, \gamma_q$ is a basis of the one-forms on U with $d\omega_i = -\omega_i \wedge \gamma_i$ and $d\gamma_i = 0$.

Recall that $\widehat{A}(\xi_1, ..., \xi_{2q}) = \prod_{j=1}^{2q} \frac{\xi_j/2}{\sinh(\xi_j/2)}$. We want to compute

$$\int_F \widehat{A}(T(S^{2p-1} \times S^1)) = \int_{S^{2q-1} \times S^1} \widehat{A}(T(S^{2p-1} \times S^1)) = \int_{S^{2q-1} \times (1,e)} \widehat{A}(\Omega).$$

As $(d\delta_i)^3 = 0$ and $(d\delta_i)^2 = 2(x_i^2 + y_i)^2 dx_i \wedge dy_i \wedge \omega_i \wedge \gamma_i$, the only term of $\widehat{A}(\Omega)$ which will be non-zero when integrated over F is, just as in [22],

$$\begin{aligned}\widehat{A}_{2q}(\Omega) &= \widehat{A}_{2q}(\lambda_1, \lambda_1, ..., \lambda_q, \lambda_q)(d(\lambda\delta))^{2q} \\ &= \widehat{A}_{2q}(\lambda_1, \lambda_1, ..., \lambda_q, \lambda_q)(2q)! \prod_{i=1}^{q} (\lambda_i^2 (x_i^2 + y_i^2) dx_i \wedge dy_i \wedge \omega_i \wedge \gamma_i,\end{aligned}$$

where $\widehat{A}_{2q}(\xi_1, ..., \xi_{2q})$ is the term in $\widehat{A}(\xi_1, ..., \xi_{2q})$ of degree $2q$. Thus,

$$\int\limits_{S^{2q-1}\times(1,e)} \widehat{A}_{2q}(\Omega) =$$

$$\widehat{A}_{2q}(\lambda_1, \lambda_1, ..., \lambda_q, \lambda_q) \left[(2q)! \int\limits_{S^{2q-1}\times(1,e)} \prod_{i=1}^{q} (\lambda_i^2 (x_i^2 + y_i^2) dx_i \wedge dy_i) \right] \prod_{i=1}^{q} (\omega_i \wedge \gamma_i) =$$

$$\frac{\pi^q (e^{4q} - 1) \widehat{A}_{2q}(\lambda_1, \lambda_1, ..., \lambda_q, \lambda_q)}{(\lambda_1 \cdots \lambda_q)^2} \prod_{i=1}^{q} (\omega_i \wedge \gamma_i),$$

by Lemma 5.8 of [22], which is a nowhere zero $2q$ form on N. Note that $\widehat{A}_{2q}(\lambda_1, \lambda_1, ..., \lambda_q, \lambda_q)$ is a non-zero constant times $(\lambda_1 \cdots \lambda_q)^2$. Thus, there is a non-zero constant C_q so that

$$\int\limits_N \int\limits_{S^{2q-1}\times(1,e)} \widehat{A}(\Omega) = \int\limits_N \int\limits_F \widehat{A}(TF) = C_q \operatorname{vol}(N).$$

Data availability

No data was used for the research described in the article.

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